The Centre-Area Lemma

From First Principles

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Contents

Preface

Who this book is for. A reader who has just finished a first year of undergraduate mathematics — Cambridge Part IA, or its equivalent. What is assumed is exactly what such a course supplies: vectors, inner products and the Cauchy–Schwarz inequality in $\mathbb{R}^n$; 2×2 determinants; sequences, limits and the ε–δ definition of continuity on the real line; the Bolzano–Weierstrass theorem in one variable; the mean value theorem; and the addition formulas for sin and cos. Everything else is built here. In particular, no topology, no convexity, no optimisation theory and no several-variable calculus is presupposed: the little that is needed of each is proved from scratch, in the plane, and in exactly the form the argument later consumes.

Where the book is going. To one sentence.

> *Let S_1, S_2, S_3 be unit squares in the plane, rotated arbitrarily and independently, with pairwise disjoint interiors, and let P_1, P_2, P_3 be their centres. If the triangle $P_1P_2P_3$ is non-obtuse, then* $\mathrm{Area}(P_1P_2P_3) \geq \frac{1}{2}$.

This is the *centre-area lemma.* It is stated as Theorem 1.1 in Chapter 1, cited by that number in every chapter in between, and proved in Chapter 13. The constant $\frac{1}{2}$ is best possible, the angle hypothesis cannot be dropped, and there is no analogue for four centres; the final chapter proves all three.

The proof is not long, but it draws on four different bodies of technique — plane geometry, convexity, constrained optimisation with duality, and a little combinatorics — and the bulk of this book is spent building those four, from a Part IA starting point, in the precise form the endgame needs. The one-sentence summary of the argument is: assume a counterexample; replace the disjointness of each pair of squares by a single linear inequality on the centres; minimise the doubled area over the resulting compact set; convert first-order optimality at the minimiser into nonnegative multipliers; and finish with

$$2D \;=\; \lambda_1 H_1 + \lambda_2 H_2 + \lambda_3 H_3 \;\geq\; 2.$$

Chapter 1 closes with a one-page map explaining which chapter builds which layer of that sentence.

How to read it. Chapter 1 states the theorem and defines every word in it, and should be read first and in full; it ends with the map just mentioned. Chapters 2–4 are the language — orientation and the quarter turn, scalar inequalities, convex sets — and Chapters 5–8 the machinery — separating branches, configuration space, compactness and first-order optimality, and the Farkas lemma. The proof proper begins at Chapter 9 and does not stop until Chapter 13.

Two reading orders work. The linear one is the intended one. The impatient one is to read Chapter 1, then Chapters 9–13, and to consult Chapters 2–8 only when a result is cited that you do not already believe: each chapter after the first opens with an explicit list of what it assumes from earlier chapters, and each result is cited by number rather than by name, so this is a practical strategy rather than a pious hope. Chapter 3, in particular, is pure scalar technique with no geometry in it, and may be deferred until Chapter 11 needs it.

Every chapter closes with exercises. They are not decoration: several of them supply constructions and side results that the main text quotes, and those are marked as such where they occur.

A standing warning, printed here and repeated wherever it applies. From Chapter 5 onwards a *branch* is a linear inequality on the centres, chosen once and then frozen. To say that a branch is *active* is to say that this chosen inequality holds with equality. It is never a statement that two squares touch. No square is moved or rotated anywhere in this book.

Disclaimer. This book was produced with a mix of ChatGPT and Claude, building on AI-generated research, and has not yet been adequately human reviewed.

Chapter 1

Squares, Interiors, and the Statement

Assumed background. Nothing from later chapters. From Part IA: inner products and the Cauchy–Schwarz inequality in $\mathbb{R}^n$, 2×2 determinants, sequences and limits of real numbers, and the cosine rule. The only new vocabulary is that of §1.2, which is developed from the norm.

This chapter states the centre-area lemma and defines the terms in it. It also proves the one-dimensional predecessor of the theorem, that two centres of interior-disjoint unit squares lie at distance at least 1, and the hypothesis-free bound that the theorem improves.

1.1 Packing unit squares and triples of centres

Let $s(n)$ denote the infimum of the sides of closed squares that contain n unit squares with pairwise disjoint interiors, rotations allowed. Trivially

$$\sqrt{n} \;\leq\; s(n) \;\leq\; \lceil \sqrt{n} \rceil :$$

the lower bound because n interior-disjoint unit squares inside a square of side s occupy total area n and so force $n \leq s^2$ (this uses additivity of area over sets with pairwise disjoint interiors, which we do not prove); and the upper bound because a closed square of side $\lceil \sqrt{n} \rceil$ is tiled in a grid by $\lceil \sqrt{n} \rceil^2 \geq n$ axis-parallel unit squares, of which one may keep any n.

The problem is discussed by Erdős and Graham (*On packing squares with equal squares*, J. Combin. Theory Ser. A **19** (1975), 119–123), who proved that the wasted area $W(s) = s^2 - \max\{n : s(n) \leq s\}$ is $O(s^{7/11})$; for the state of the art see E. Friedman's dynamic survey (*Packing unit squares in squares: a survey and new results*, Electron. J. Combin., Dynamic Survey DS7, version of 2009, doi:10.37236/28). Nothing from the packing literature is used below; the packing problem serves only as motivation.

Lower bounds for $s(n)$ are counting arguments over the *centres* of the packed squares. The pairwise constraint is one-dimensional: two centres lie at distance at least 1, which we prove in §1.6. The first two-dimensional constraint concerns triples of centres. Its quantitative form is a lower bound on the area of the triangle spanned by three centres, and the sharp such bound is the following.

Theorem 1.1 (Centre-area lemma)**.** *Let S_1, S_2, S_3 be unit squares in the plane, rotated arbitrarily and independently, with pairwise disjoint interiors, and let P_1, P_2, P_3 be their centres. If the triangle $P_1P_2P_3$ is non-obtuse, then*

$$\mathrm{Area}(P_1P_2P_3) \;\geq\; \tfrac{1}{2}.$$

The terms of Theorem 1.1 are defined in §§1.2–1.5, and its proof is assembled in Chapter 13. It forces $wh \geq 1$ whenever three centres of interior-disjoint unit squares lie in a closed $w \times h$ rectangle and their centre triangle is non-obtuse (Exercise 13.6).

The letter s is used for a packing function only in this section; from Chapter 2 onwards, s (with or without a subscript) denotes a sine.

1.2 Balls, open and closed sets, interiors

This section fixes the vocabulary needed to speak of disjoint interiors, together with two facts about linear inequalities: strict ones survive small perturbations, and non-strict ones cut out closed sets. Everything is built from the norm and Cauchy–Schwarz. The general theory is developed in Chapter 7.

Definition 1.2 (Balls, open sets, closed sets, interiors)**.** For $x \in \mathbb{R}^n$ and $\rho > 0$, the *open ball* of radius ρ about x is

$$B(x, \rho) = \{\, y \in \mathbb{R}^n : \ \|y - x\| < \rho \,\}.$$

A set $U \subseteq \mathbb{R}^n$ is *open* if for every $x \in U$ there is $\rho > 0$ with $B(x, \rho) \subseteq U$. A set is *closed* if its complement is open. The *interior* of $X \subseteq \mathbb{R}^n$ is

$$\operatorname{int} X = \{\, x \in \mathbb{R}^n : \ B(x, \rho) \subseteq X \text{ for some } \rho > 0 \,\}.$$

Three consequences are immediate: $\operatorname{int} X \subseteq X$, since $x \in B(x, \rho)$; a set X is open if and only if $X = \operatorname{int} X$, by comparing the two definitions; and $\operatorname{int} X$ is itself open, because if $B(x, \rho) \subseteq X$ then every $y \in B(x, \rho)$ has a ball about it inside $B(x, \rho)$, by Proposition 1.3 below.

Proposition 1.3. *Every open ball is open.*

Proof. Let $y \in B(x, \rho)$ and put $\rho' = \rho - \|y - x\| > 0$. If $z \in B(y, \rho')$ then, by the triangle inequality,

$$\|z - x\| \ \le \ \|z - y\| + \|y - x\| \ < \ \rho' + \|y - x\| \ = \ \rho,$$

so $z \in B(x, \rho)$. Hence $B(y, \rho') \subseteq B(x, \rho)$. □

Lemma 1.4 (Cauchy–Schwarz; Part IA)**.** *For $u, v \in \mathbb{R}^n$ we have $|\langle u, v\rangle| \le \|u\| \, \|v\|$, with equality if and only if u and v are linearly dependent.*

We take this from Part IA without proof.

Lemma 1.5 (Strict linear inequalities are stable)**.** *Let $u \in \mathbb{R}^n$ be nonzero, let $c \in \mathbb{R}$, and suppose that $\langle x_0, u\rangle < c$. Put $\rho = \big(c - \langle x_0, u\rangle\big)/\|u\| > 0$. Then $\langle y, u\rangle < c$ for every $y \in B(x_0, \rho)$.*

Proof. For $y \in B(x_0, \rho)$,

$$\langle y, u\rangle = \langle x_0, u\rangle + \langle y - x_0, u\rangle \ \le \ \langle x_0, u\rangle + \|y - x_0\| \, \|u\| \ < \ \langle x_0, u\rangle + \rho\|u\| = c,$$

using Lemma 1.4 in the middle step. □

Proposition 1.6 (Slab sets)**.** *Let $u_1, \dots, u_k \in \mathbb{R}^n$ be nonzero and let $c_1, \dots, c_k \in \mathbb{R}$. Then*

(i) $U = \{x : \ \langle x, u_i\rangle < c_i$ for $i = 1, \dots, k\}$ is open;

(ii) $V = \{x : \ \langle x, u_i\rangle \le c_i$ for $i = 1, \dots, k\}$ is closed.

Proof. (i) Let $x_0 \in U$ and let $\rho_i > 0$ be the radius supplied by Lemma 1.5 for the ith inequality. Put $\rho = \min\{\rho_1, \dots, \rho_k\}$, which is positive because the index set is finite; then $B(x_0, \rho) \subseteq U$.

(ii) Let $x_0 \notin V$, so $\langle x_0, u_i\rangle > c_i$ for some i, that is, $\langle x_0, -u_i\rangle < -c_i$. By Lemma 1.5 applied to $-u_i$ there is $\rho > 0$ with $\langle y, -u_i\rangle < -c_i$, i.e. $\langle y, u_i\rangle > c_i$, for all $y \in B(x_0, \rho)$; so $B(x_0, \rho)$ misses V. Hence the complement of V is open. □

Convergence in $\mathbb{R}^n$ is defined exactly as in $\mathbb{R}$: we write $x_k \to x$ to mean $\|x_k - x\| \to 0$ as $k \to \infty$.

Proposition 1.7 (Sequential characterisation of closedness)**.** *A set $X \subseteq \mathbb{R}^n$ is closed if and only if: whenever (x_k) is a sequence in X with $x_k \to x$ for some $x \in \mathbb{R}^n$, we have $x \in X$.*

Proof. ($\Rightarrow$) Let X be closed, let $x_k \in X$ with $x_k \to x$, and suppose $x \notin X$. Since the complement of X is open there is $\rho > 0$ with $B(x,\rho) \cap X = \emptyset$. But $\|x_k - x\| < \rho$ for all large k, so $x_k \in B(x,\rho) \cap X$, a contradiction.

($\Leftarrow$) We prove the contrapositive. Suppose X is not closed, so the complement of X is not open: there is $x \notin X$ such that no ball about x is contained in the complement, i.e. $B(x, 1/k) \cap X \neq \emptyset$ for every $k \geq 1$. Choose $x_k \in B(x, 1/k) \cap X$. Then $\|x_k - x\| < 1/k \to 0$, so $x_k \to x$ with $x_k \in X$ and $x \notin X$, and the stated condition fails. $\square$

This is the form in which closedness is used in Chapter 7; it is also why the constraints below are given by non-strict polynomial inequalities, which pass to limits.

Example 1.8 (Interiors and unions)**.** In $\mathbb{R}$ we have $\operatorname{int}[0,1] = (0,1)$ and $\operatorname{int}[1,2] = (1,2)$, while

$$\operatorname{int}\big([0,1] \cup [1,2]\big) = \operatorname{int}[0,2] = (0,2),$$

which strictly contains $(0,1) \cup (1,2)$: the point 1 lies in the interior of the union but in neither interior. So interiors do not commute with unions.

Disjointness of two sets implies disjointness of their interiors, since $\operatorname{int} X \subseteq X$; the converse fails, as Example 1.13(a) will show. "Disjoint interiors" is therefore the strictly weaker hypothesis, and it is the one Theorem 1.1 uses.

1.3 Unit squares in any orientation

A rotated square is most conveniently described as the intersection of two slabs, one per axis direction. This is the description used throughout, and it is generalised in Chapter 4.

Definition 1.9 (Frame, unit square)**.** A *frame* is an ordered orthonormal pair $F = (e, f)$ of vectors in $\mathbb{R}^2$; the four unit vectors $\pm e, \pm f$ are its *axes*. For a point $P \in \mathbb{R}^2$, the *unit square with frame F and centre P* is

$$Q(F,P) = \Big\{ X \in \mathbb{R}^2 : \ \big|\langle X - P, e\rangle\big| \leq \tfrac{1}{2} \ \text{ and } \ \big|\langle X - P, f\rangle\big| \leq \tfrac{1}{2} \Big\}.$$

Proposition 1.10 (Vertices, area, and interior of $Q(F,P)$)**.** *Let $F = (e,f)$ be a frame and $P \in \mathbb{R}^2$.*

(i) $Q(F,P)$ is the closed square with vertices $P \pm \frac{1}{2}e \pm \frac{1}{2}f$; it has side 1 and area 1, and it is closed and bounded.

(ii) $\operatorname{int} Q(F,P) = \big\{X : \ |\langle X - P, e\rangle| < \frac{1}{2}$ *and* $|\langle X - P, f\rangle| < \frac{1}{2}\big\}$.

Proof. (i) Since (e,f) is an orthonormal basis of $\mathbb{R}^2$, every X can be written uniquely as $X = P + ae + bf$, and then $a = \langle X - P, e\rangle$ and $b = \langle X - P, f\rangle$. The map $\Phi(a,b) = P + ae + bf$ satisfies

$$\|\Phi(a,b) - \Phi(a',b')\|^2 = \|(a-a')e + (b-b')f\|^2 = (a-a')^2 + (b-b')^2,$$

by orthonormality, so Φ is an isometry of $\mathbb{R}^2$ onto $\mathbb{R}^2$. The defining conditions read $|a| \leq \frac{1}{2}$, $|b| \leq \frac{1}{2}$, so $Q(F,P) = \Phi\big([-\frac{1}{2}, \frac{1}{2}]^2\big)$ is an isometric copy of a closed axis-parallel square of side 1, with the four stated vertices; area and side are preserved by isometries, and $Q(F,P) \subseteq B(P,1)$ is bounded. Closedness is Proposition 1.6(ii) applied to the four inequalities $\langle X, \pm e\rangle \leq \frac{1}{2} \pm \langle P, e\rangle$ and $\langle X, \pm f\rangle \leq \frac{1}{2} \pm \langle P, f\rangle$.

(ii) Write U for the set on the right-hand side. If $X \in U$ then the four strict inequalities $\langle X, \pm e\rangle < \frac{1}{2} \pm \langle P, e\rangle$, $\langle X, \pm f\rangle < \frac{1}{2} \pm \langle P, f\rangle$ hold, so by Proposition 1.6(i) there is a ball about X on which they all hold; that ball lies in $Q(F,P)$, whence $X \in \operatorname{int} Q(F,P)$. Conversely suppose $X \notin U$, say $|\langle X - P, e\rangle| \geq \frac{1}{2}$. If the inequality is strict then $X \notin Q(F,P)$, so $X \notin \operatorname{int} Q(F,P)$. If $\langle X - P, e\rangle = \frac{1}{2}$ (the case $-\frac{1}{2}$ is symmetric), then for every $\rho > 0$ the point $X + \frac{\rho}{2}e$ lies in $B(X,\rho)$ and satisfies $\langle X + \frac{\rho}{2}e - P, e\rangle = \frac{1}{2} + \frac{\rho}{2} > \frac{1}{2}$, so it is not in $Q(F,P)$. Hence no ball about X lies in $Q(F,P)$. $\square$

Corollary 1.11. *If $X \in \operatorname{int} Q(F,P)$ then $|\langle X - P, e\rangle| < \frac{1}{2}$ and $|\langle X - P, f\rangle| < \frac{1}{2}$: a strict inequality on each axis separately.*

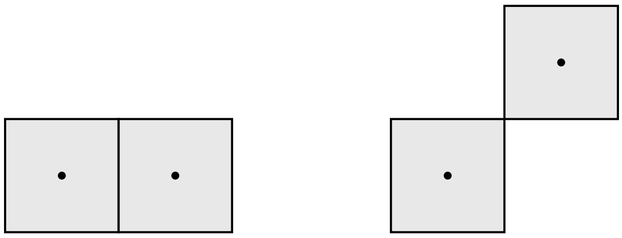


Figure 1.1: Two admissible pairs of axis-parallel unit squares (interiors shaded, boundaries solid, centres marked). Neither pair is disjoint as a pair of sets: the left pair meets in a segment of length 1, the right pair in a single point. In both cases the interiors are disjoint.

The branch arguments of Chapter 5 apply this corollary one axis direction at a time.

Definition 1.12 (Unit square, admissible family)**.** A *unit square* is a set of the form $Q(F, P)$; the point P is its *centre* and F is *a* frame for it. A finite family of unit squares is *admissible* if their interiors are pairwise disjoint.

The frame is not unique: eight frames F' satisfy $Q(F', P) = Q(F, P)$, since one may choose which axis direction is listed first and the signs of both. The centre is unique. Every closed square of side 1 in the plane is of the form $Q(F, P)$, obtained by taking e and f to be unit vectors along two adjacent edge directions. These statements are set as Exercise 1.2.

Example 1.13 (Disjoint interiors, not disjoint sets)**.** Let $F = \big((1, 0), (0, 1)\big)$ be the axis-parallel frame, so $Q(F, P)$ is the axis-parallel closed square of side 1 about P.

(a) $Q(F, (0, 0))$ and $Q(F, (1, 0))$ meet exactly in the segment $\{\frac{1}{2}\} \times [-\frac{1}{2}, \frac{1}{2}]$, of length 1. Their interiors are disjoint: a common interior point $X = (x, y)$ would satisfy $x < \frac{1}{2}$ from the first square and $x - 1 > -\frac{1}{2}$, i.e. $x > \frac{1}{2}$, from the second.

(b) $Q(F, (0, 0))$ and $Q(F, (1, 1))$ meet exactly in the single point $(\frac{1}{2}, \frac{1}{2})$; the same computation on either axis shows the interiors are disjoint.

(c) $Q(F, (0, 0))$ and $Q(F, (2, 0))$ are disjoint as sets.

Admissibility therefore permits squares to share an edge and to share a corner, and the extremal configuration of Theorem 1.1 uses both (Figure 1.3).

Call a set $X \subseteq \mathbb{R}^n$ *convex* if for all $x, y \in X$ and all $t \in [0, 1]$ the point $(1 - t)x + ty$ lies in X; in particular the midpoint $\frac{1}{2}(x + y)$ of any two of its points lies in it (see Chapter 4).

Example 1.14 (Admissibility is not a convex condition)**.** Keep F axis-parallel and fix the square $Q(F, (0, 0))$. The pairs $\big(Q(F, (0, 0)), Q(F, (1, 0))\big)$ and $\big(Q(F, (0, 0)), Q(F, (-1, 0))\big)$ are both admissible, by Example 1.13(a) and its reflection. The midpoint of the two second centres is $(0, 0)$, and $\big(Q(F, (0, 0)), Q(F, (0, 0))\big)$ is not admissible. Hence the set of admissible second centres is not convex, and in particular it is not the solution set of any family of linear inequalities.

Because admissibility of a pair is not a convex condition, Chapter 5 replaces it, for each pair, by a single linear inequality sufficient for admissibility; the choice is fixed in Chapter 9.

Example 1.15 (A tilted interior computation)**.** Let $e = (\frac{3}{5}, \frac{4}{5})$, $f = (-\frac{4}{5}, \frac{3}{5})$ — an orthonormal pair, since $\|e\| = \|f\| = 1$ and $\langle e, f\rangle = -\frac{12}{25} + \frac{12}{25} = 0$ — and let $P = (0, 0)$. Take $X = (\frac{1}{2}, \frac{1}{4})$. Then

$$\langle X, e\rangle = \tfrac{3}{10} + \tfrac{1}{5} = \tfrac{1}{2}, \qquad \langle X, f\rangle = -\tfrac{2}{5} + \tfrac{3}{20} = -\tfrac{1}{4}.$$

So $X \in Q(F, 0)$ but, by Proposition 1.10(ii), $X \notin \operatorname{int} Q(F, 0)$: the point lies on an edge, which is not evident from the coordinates. Interiority is decided by the two slab inequalities.

1.4 The determinant as signed area

We record the standard facts relating the determinant to area, in the form of an area functional of three vertices.

Definition 1.16. For $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $\mathbb{R}^2$,

$$\det(u, v) = u_1 v_2 - u_2 v_1.$$

The formula gives at once that det is bilinear (each of $u_1 v_2$ and $u_2 v_1$ is linear in u for fixed v and conversely); it is alternating, since $\det(u, u) = u_1 u_2 - u_2 u_1 = 0$ and $\det(v, u) = -\det(u, v)$; and $\det\big((1, 0), (0, 1)\big) = 1 \cdot 1 - 0 \cdot 0 = 1$.

Lemma 1.17 (Lagrange's identity in the plane)**.** *For all $u, v \in \mathbb{R}^2$,*

$$\det(u, v)^2 + \langle u, v\rangle^2 = \|u\|^2 \, \|v\|^2.$$

Proof. Expand:

$$\begin{aligned}(u_1 v_2 - u_2 v_1)^2 + (u_1 v_1 + u_2 v_2)^2 &= u_1^2 v_2^2 - 2u_1 v_2 u_2 v_1 + u_2^2 v_1^2 + u_1^2 v_1^2 + 2u_1 v_1 u_2 v_2 + u_2^2 v_2^2 \\ &= u_1^2 (v_1^2 + v_2^2) + u_2^2 (v_1^2 + v_2^2) = \|u\|^2 \|v\|^2.\end{aligned}$$

□

This identity reappears in Chapter 2 in the unit-vector form $\langle n, m\rangle^2 + \det(n, m)^2 = 1$, and in Chapter 5 in the computation of the projected half-width.

Corollary 1.18 (Geometric reading)**.** *Let $u, v \in \mathbb{R}^2$ be nonzero and let $\theta \in [0, \pi]$ be the angle between them, defined by $\cos\theta = \langle u, v\rangle/(\|u\|\|v\|)$ (legitimate, since the quotient lies in $[-1, 1]$ by Lemma 1.4 and* cos *is a decreasing bijection $[0, \pi] \to [-1, 1]$). Then*

$$|\det(u, v)| = \|u\| \, \|v\| \, \sin\theta,$$

and $\det(u, v) = 0$ if and only if u, v are linearly dependent.

Proof. By Lemma 1.17,

$$\det(u, v)^2 = \|u\|^2 \|v\|^2 - \langle u, v\rangle^2 = \|u\|^2 \|v\|^2 \big(1 - \cos^2\theta\big) = \big(\|u\|\|v\| \sin\theta\big)^2,$$

and $\sin\theta \geq 0$ on $[0, \pi]$, so taking nonnegative square roots gives the first claim. For the second, $\det(u, v) = 0$ forces $\langle u, v\rangle^2 = \|u\|^2 \|v\|^2$, which is the equality case of Lemma 1.4, hence linear dependence; conversely $v = \lambda u$ gives $\det(u, v) = \lambda \det(u, u) = 0$, and the case $u = 0$ or $v = 0$ is trivial. □

Corollary 1.19 (Distance to a line)**.** *Let $u \in \mathbb{R}^2$ be nonzero. The distance from v to the line $\mathbb{R}u = \{tu : t \in \mathbb{R}\}$ equals $|\det(u, v)|/\|u\|$.*

Proof. Write $v = \alpha u + v^\perp$ with $\alpha = \langle u, v\rangle/\|u\|^2$ and $v^\perp = v - \alpha u$; then $\langle u, v^\perp\rangle = \langle u, v\rangle - \alpha\|u\|^2 = 0$. For any t,

$$\|v - tu\|^2 = \|v^\perp + (\alpha - t)u\|^2 = \|v^\perp\|^2 + (\alpha - t)^2 \|u\|^2 \ \geq\ \|v^\perp\|^2,$$

with equality at $t = \alpha$; so the distance is $\|v^\perp\|$. Finally

$$\|v^\perp\|^2 = \|v\|^2 - \frac{\langle u, v\rangle^2}{\|u\|^2} = \frac{\|u\|^2 \|v\|^2 - \langle u, v\rangle^2}{\|u\|^2} = \frac{\det(u, v)^2}{\|u\|^2}$$

by Lemma 1.17. □

Corollary 1.19 justifies the formula "base times height".

Definition 1.20 (Area of a triangle)**.** For points $X, Y, Z \in \mathbb{R}^2$,

$$\operatorname{Area}(XYZ) = \tfrac{1}{2}\big|\det(Y - X,\ Z - X)\big|.$$

Proposition 1.21. *Let $X, Y, Z \in \mathbb{R}^2$.*

(i) If $X \neq Y$*, then* $\mathrm{Area}(XYZ) = \frac{1}{2}\|Y - X\| \cdot h$*, where* h *is the distance from* Z *to the line through* X *and* Y*.*

(ii) $\det(Y - X, Z - X) = \det(Z - Y, X - Y) = \det(X - Z, Y - Z)$*; so Definition 1.20 does not depend on which vertex is listed first, and transposing two of the three points reverses the sign of the determinant.*

(iii) (Shoelace form) $\det(Y - X, Z - X) = \det(X, Y) + \det(Y, Z) + \det(Z, X)$*.*

Proof. (i) Put $u = Y - X$ and $v = Z - X$. The line through X and Y is $X + \mathbb{R}u$, so the distance from Z to it is the distance from v to $\mathbb{R}u$, which is $|\det(u, v)|/\|u\|$ by Corollary 1.19. Hence $\frac{1}{2}\|u\|h = \frac{1}{2}|\det(u, v)|$.

(iii) By bilinearity and alternation,

$$\begin{aligned}\det(Y - X, Z - X) &= \det(Y, Z) - \det(Y, X) - \det(X, Z) + \det(X, X)\\ &= \det(X, Y) + \det(Y, Z) + \det(Z, X).\end{aligned}$$

(ii) The right-hand side of (iii) is invariant under the cyclic relabelling $X \mapsto Y \mapsto Z \mapsto X$, since it is the sum of det over consecutive pairs in cyclic order; applying (iii) to the relabelled triples gives the two stated equalities. A transposition, say swapping X and Y, sends the sum $\det(X, Y) + \det(Y, Z) + \det(Z, X)$ to $\det(Y, X) + \det(X, Z) + \det(Z, Y)$, which is its negative by alternation. □

Remark 1.22 (The signed area)*.* As a function of the six coordinates of X, Y, Z, the quantity $\det(Y - X, Z - X)$ is a polynomial, hence infinitely differentiable. Its absolute value is not differentiable where the determinant vanishes, that is, by Corollary 1.18, at collinear triples. The optimisation below is therefore carried out on the signed quantity, with the sign fixed in advance. Accordingly, fix a sign $\sigma \in \{-1, +1\}$ and set

$$D \;=\; \sigma \det(P_2 - P_1,\, P_3 - P_1).$$

We call D the *doubled area.* When σ is chosen so that $D \geq 0$ we have $D = 2\,\mathrm{Area}(P_1P_2P_3)$, and the target inequality $\mathrm{Area} \geq \frac{1}{2}$ of Theorem 1.1 reads

$$D \;\geq\; 1.$$

From Chapter 9 onwards σ is computed once, at an assumed counterexample, and thereafter treated as a constant. Chapter 2 develops the calculus of the signed area.

Example 1.23 (The two running configurations)**.** (a) $P_1 = (0, 0)$, $P_2 = (1, 0)$, $P_3 = (0, 1)$: here $\det\big((1, 0), (0, 1)\big) = 1 \cdot 1 - 0 \cdot 0 = 1$, so $\mathrm{Area} = \frac{1}{2}$, and $D = 1$ with $\sigma = +1$.

(b) $P_1 = (0, 0)$, $P_2 = (1, \varepsilon)$, $P_3 = (2, 0)$: here $\det\big((1, \varepsilon), (2, 0)\big) = 1 \cdot 0 - \varepsilon \cdot 2 = -2\varepsilon$, so $\mathrm{Area} = \varepsilon$, and $D = 2\varepsilon$ with $\sigma = -1$.

Both configurations return in §1.7.

1.5 Non-obtuseness

The angle hypothesis of Theorem 1.1 has three equivalent forms, used interchangeably below.

Definition 1.24 (Angle at a vertex)**.** Let $X, Y, Z \in \mathbb{R}^2$ be pairwise distinct. The *angle at* X is the unique $\theta_X \in [0, \pi]$ with

$$\cos\theta_X = \frac{\langle Y - X,\, Z - X\rangle}{\|Y - X\|\,\|Z - X\|}.$$

This is well defined: the quotient lies in $[-1, 1]$ by Lemma 1.4, and $\cos : [0, \pi] \to [-1, 1]$ is a decreasing bijection.

The definition asks only for pairwise distinctness, not for noncollinearity; a collinear triple has angles 0 or π. The condition is applied below to triples not yet known to be noncollinear.

Proposition 1.25 (Three equivalent forms)**.** *Let $X, Y, Z \in \mathbb{R}^2$ be pairwise distinct. The following are equivalent:*

(i) $\theta_X \le \pi/2$*;*

(ii) $\langle Y - X,\, Z - X\rangle \ge 0$*;*

(iii) $\|Y - Z\|^2 \le \|X - Y\|^2 + \|X - Z\|^2$*.*

The same holds with all three inequalities made strict simultaneously.

Proof. (i)⇔(ii): cos is nonnegative on $[0, \pi]$ exactly on $[0, \pi/2]$, and the denominator $\|Y - X\|\|Z - X\|$ is strictly positive, so $\cos\theta_X \ge 0$ if and only if $\langle Y - X, Z - X\rangle \ge 0$.

(ii)⇔(iii): put $u = Y - X$, $v = Z - X$, so $Y - Z = u - v$ and

$$\|Y - Z\|^2 = \|u - v\|^2 = \|u\|^2 + \|v\|^2 - 2\langle u, v\rangle = \|X - Y\|^2 + \|X - Z\|^2 - 2\langle u, v\rangle. \tag{1.1}$$

(Substituting $\langle u, v\rangle = \|u\|\|v\| \cos\theta_X$ turns (1.1) into the cosine rule.) Thus (iii) holds if and only if $-2\langle u, v\rangle \le 0$, which is (ii). Strictness corresponds throughout. □

Definition 1.26. A triple of pairwise distinct points is *non-obtuse* if the equivalent conditions of Proposition 1.25 hold at each of its three vertices, and *acute* if the strict versions hold at each vertex. A triangle is non-obtuse, respectively acute, accordingly.

Remark 1.27 (Form (iii)). Form (iii) is a polynomial inequality in the coordinates of the points, involving no square roots, no division and no inverse trigonometric functions. The feasible set of the minimisation carried out in Chapter 9 is therefore cut out by finitely many non-strict polynomial inequalities, and is closed, by Proposition 1.7 and the general theory of Chapter 7.

Lemma 1.28 (Non-obtuseness excludes collinearity)**.** *Three pairwise distinct collinear points are not non-obtuse.*

Proof. Write the points as $p + ad$, $p + bd$, $p + cd$ with $d \ne 0$ and a, b, c distinct reals. Relabelling (the hypothesis and the conclusion are symmetric in the three points), take the middle point to be $Y = p + bd$ with $a < b < c$, and set $X = p + ad$, $Z = p + cd$. Then $X - Y = (a - b)d$ and $Z - Y = (c - b)d$ with $a - b < 0 < c - b$, so

$$\langle X - Y,\, Z - Y\rangle = (a - b)(c - b)\|d\|^2 < 0,$$

and form (ii) of Proposition 1.25 fails at Y. (Indeed $\theta_Y = \pi$.) □

Corollary 1.29. *If P_1, P_2, P_3 are pairwise distinct and non-obtuse, then $\det(P_2 - P_1, P_3 - P_1) \ne 0$; hence with σ chosen as its sign we have $D > 0$.*

Proof. If the determinant vanished, then $P_2 - P_1$ and $P_3 - P_1$ would be linearly dependent by Corollary 1.18, and since both are nonzero the three points would be collinear, contradicting Lemma 1.28. □

Example 1.30 (Checking the two running configurations)**.** (a) *The right isosceles triple.* The points $(0, 0)$, $(1, 0)$ and $(0, 1)$ have squared side lengths 1, 1 and 2, so the three instances of form (iii) read $2 \le 1 + 1$, $1 \le 1 + 2$ and $1 \le 1 + 2$. All hold, so the triple is non-obtuse; the first holds with equality, so the triangle is right-angled at the origin and is not acute.

(b) $(0, 0), (1, \varepsilon), (2, 0)$ with $0 < \varepsilon < 1$. At the middle vertex $(1, \varepsilon)$ the two edge vectors are $(-1, -\varepsilon)$ and $(1, -\varepsilon)$, and

$$\big\langle (-1, -\varepsilon), (1, -\varepsilon)\big\rangle = \varepsilon^2 - 1 < 0,$$

so form (ii) fails there: the triple is obtuse. Equivalently, in form (iii), the squared length 4 of the opposite side exceeds $(1 + \varepsilon^2) + (1 + \varepsilon^2) = 2 + 2\varepsilon^2$, since $4 - (2 + 2\varepsilon^2) = 2(1 - \varepsilon^2) > 0$.

1.6 Centre separation

We prove the one-dimensional predecessor of Theorem 1.1, and deduce the hypothesis-free bound.

Lemma 1.31 (Inscribed disc)**.** *For every frame F and point P, $B(P, \frac{1}{2}) \subseteq \operatorname{int} Q(F, P)$.*

Proof. Let $\|X - P\| < \frac{1}{2}$. By Lemma 1.4,

$$|\langle X - P, e\rangle| \le \|X - P\|\,\|e\| = \|X - P\| < \tfrac{1}{2},$$

and likewise with f in place of e. Now apply Proposition 1.10(ii). □

The radius is optimal: $P + \frac{1}{2}e$ lies at distance exactly $\frac{1}{2}$ from P and satisfies $\langle X - P, e\rangle = \frac{1}{2}$, so it is not an interior point.

Lemma 1.32 (Centre separation)**.** *The centres of two unit squares with disjoint interiors are at distance at least* 1*.*

Proof. Let the squares be $Q(F, P)$ and $Q(G, R)$ and suppose, for a contradiction, that $\|R-P\| < 1$. Put $M = \frac{1}{2}(P + R)$. Then

$$\|M - P\| = \tfrac{1}{2}\|R - P\| < \tfrac{1}{2}, \qquad \|M - R\| = \tfrac{1}{2}\|P - R\| < \tfrac{1}{2},$$

so by Lemma 1.31 the point M lies in $\operatorname{int} Q(F, P)$ and in $\operatorname{int} Q(G, R)$. This contradicts disjointness of the interiors. □

Remark 1.33. (a) The bound is attained: the two axis-parallel squares of Example 1.13(a), with centres $(0, 0)$ and $(1, 0)$, have interiors disjoint and centres at distance 1.

(b) The argument uses only the shadow of each square on the line through the two centres, and the inscribed disc says that every such shadow has half-length at least $\frac{1}{2}$. Chapter 5 states this as the bound $w_F(n) \ge \frac{1}{2}$ on the projected half-width of a frame, for every unit vector n.

(c) To contradict disjointness of two interiors one exhibits a single point lying in both; the midpoint of the two centres is the natural candidate. We refer to this below as the *midpoint trick*.

Example 1.34 (Separation for two frames at $\pi/4$)**.** When the two orientations differ, Lemma 1.32 is not sharp. Let $F = \big((1, 0), (0, 1)\big)$ be axis-parallel and let $G = \big(e', f'\big)$ with $e' = \frac{\sqrt{2}}{2}(1, 1)$ and $f' = \frac{\sqrt{2}}{2}(-1, 1)$, a frame turned through $\pi/4$. Put $P = (0, 0)$ and $R = (c, 0)$ with $c > 0$ (Figure 1.2).

A point $X = (x, y)$ lies in $\operatorname{int} Q(F, P)$ if and only if $|x| < \frac{1}{2}$ and $|y| < \frac{1}{2}$. For the tilted square, $\langle X - R, e'\rangle = \frac{\sqrt{2}}{2}\big((x - c) + y\big)$ and $\langle X - R, f'\rangle = \frac{\sqrt{2}}{2}\big(-(x - c) + y\big)$, so on the horizontal axis $y = 0$ both have absolute value $\frac{\sqrt{2}}{2}|x - c|$, and $(x, 0) \in \operatorname{int} Q(G, R)$ if and only if $|x - c| < \frac{\sqrt{2}}{2}$.

Hence, if $c < \frac{1}{2} + \frac{\sqrt{2}}{2} = \frac{1+\sqrt{2}}{2}$, the open intervals $(-\frac{1}{2}, \frac{1}{2})$ and $\big(c - \frac{\sqrt{2}}{2}, c + \frac{\sqrt{2}}{2}\big)$ overlap, and any x in the overlap gives a point $(x, 0)$ in both interiors: the pair is not admissible. Conversely, if $c \ge \frac{1+\sqrt{2}}{2}$ then the pair is admissible: a common interior point X would satisfy $|\langle X - P, (1, 0)\rangle| < \frac{1}{2}$, i.e. $x < \frac{1}{2}$, while from $|\langle X - R, e'\rangle| < \frac{1}{2}$ and $|\langle X - R, f'\rangle| < \frac{1}{2}$ we get, adding,

$$\sqrt{2}\,|x - c| = \big|\langle X - R, e'\rangle - \langle X - R, f'\rangle\big| \;\le\; |\langle X - R, e'\rangle| + |\langle X - R, f'\rangle| < 1,$$

so $x > c - \frac{\sqrt{2}}{2} \ge \frac{1}{2}$, a contradiction.

Thus for this pair of orientations, with centres on the horizontal axis, the exact separation threshold is

$$\frac{1 + \sqrt{2}}{2} = 1.2071\ldots,$$

not 1. Chapter 5 computes the general threshold, $H(t) = (1 + \cos t + \sin t)/2 \ge 1$, where $t = d_4(F, G) \in [0, \pi/4]$ is the *frame distance*, defined there as the least angle between an axis of F and an axis of G; Lemma 1.32 is the case $H = 1$. Here $t = \pi/4$ and $H(\pi/4) = (1+\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2})/2 = (1+\sqrt{2})/2$, as computed.

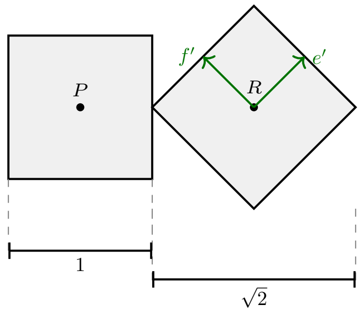


Figure 1.2: Example 1.34: an axis-parallel unit square and one turned through $\pi/4$, with centres on the horizontal axis at the minimal admissible separation $c = (1+\sqrt{2})/2$. The shadows on that axis (below, with grey dashed guides) have lengths 1 and $\sqrt{2}$, hence half-lengths $\frac{1}{2}$ and $\frac{\sqrt{2}}{2}$, whose sum is c: the two shadows touch at this separation and overlap below it. Frame axes of the tilted square are drawn in green.

Proposition 1.35 (All-angle bound)**.** *Let S_1, S_2, S_3 be unit squares with pairwise disjoint interiors and centres P_1, P_2, P_3. Then for every angle θ of the centre triangle,*

$$\mathrm{Area}(P_1P_2P_3) \;\geq\; \tfrac{1}{2}\sin\theta.$$

Proof. By Lemma 1.32 the three centres are pairwise at distance at least 1; in particular they are pairwise distinct, so all three angles are defined. Let θ be the angle at some vertex and let $a, b \geq 1$ be the lengths of the two sides adjacent to that vertex. The two edge vectors u, v at that vertex are nonzero, so Corollary 1.18 applies to them. Combining it with Definition 1.20 and Proposition 1.21(ii),

$$\mathrm{Area}(P_1P_2P_3) = \tfrac{1}{2}|\det(u,v)| = \tfrac{1}{2}\,ab\sin\theta \;\geq\; \tfrac{1}{2}\sin\theta. \qquad \square$$

Theorem 1.1 asserts that under the non-obtuseness hypothesis the factor $\sin\theta$ may be deleted; this is stronger whenever no angle of the triangle is a right angle. The constant $\frac{1}{2}$ in Proposition 1.35 is itself best possible (Exercise 1.6).

1.7 Theorem 1.1 and its hypotheses

Theorem 1.1 (Centre-area lemma; restated). *Let S_1, S_2, S_3 be unit squares in the plane, rotated arbitrarily and independently, with pairwise disjoint interiors, and let P_1, P_2, P_3 be their centres. If the triangle $P_1P_2P_3$ is non-obtuse, then*

$$\mathrm{Area}(P_1P_2P_3) \;\geq\; \tfrac{1}{2}.$$

“Unit square” and “pairwise disjoint interiors” are Definitions 1.9 and 1.12, the interiors being computed by Proposition 1.10(ii); “non-obtuse” is Definition 1.26, in any of the three forms of Proposition 1.25; “area” is Definition 1.20. The pairwise distinctness required by Definition 1.24 holds here by Lemma 1.32, which puts the centres at distance at least 1. In terms of the doubled area D of Remark 1.22, the target is $D \geq 1$.

We now examine the hypotheses in turn.

1. The constant is attained

Proposition 1.36 (Equality)**.** *The constant $\frac{1}{2}$ in Theorem 1.1 is attained.*

Proof. Take the axis-parallel frame $F = \big((1,0),(0,1)\big)$ and the three squares $Q(F,(0,0))$, $Q(F,(1,0))$, $Q(F,(0,1))$ (Figure 1.3). Write $X = (x,y)$ for a hypothetical common interior point of two of them; by Proposition 1.10(ii) membership of $\operatorname{int} Q(F,(p,q))$ means $|x-p| < \frac{1}{2}$ and $|y-q| < \frac{1}{2}$.

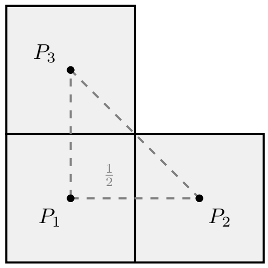


Figure 1.3: The equality configuration of Proposition 1.36: axis-parallel unit squares centred at $(0,0)$, $(1,0)$, $(0,1)$. The first two share an edge, the first and third share an edge, the second and third share only the corner $(\frac{1}{2},\frac{1}{2})$. The centre triangle (grey dashed) is right isosceles of area exactly $\frac{1}{2}$.

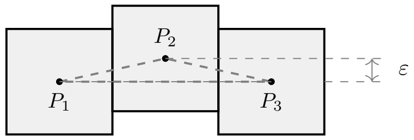


Figure 1.4: The obtuse degeneration of Example 1.37 (ε exaggerated). The squares are admissible because their horizontal shadows have pairwise disjoint interiors; the centre triangle (grey dashed) has area ε and is obtuse at P_2.

- First and second: $|x| < \frac{1}{2}$ and $|x-1| < \frac{1}{2}$ give $x < \frac{1}{2}$ and $x > \frac{1}{2}$, impossible. The two squares meet exactly in the segment $\{\frac{1}{2}\} \times [-\frac{1}{2},\frac{1}{2}]$.
- First and third: $|y| < \frac{1}{2}$ and $|y-1| < \frac{1}{2}$ are likewise incompatible. The two squares meet exactly in the segment $[-\frac{1}{2},\frac{1}{2}] \times \{\frac{1}{2}\}$.
- Second and third: $|x-1| < \frac{1}{2}$ and $|x| < \frac{1}{2}$ are incompatible (as in the first case). The two squares meet exactly in the corner $(\frac{1}{2},\frac{1}{2})$.

So the interiors are pairwise disjoint and the family is admissible.

The centre triangle has vertices $(0,0),(1,0),(0,1)$. By Example 1.30(a) it is non-obtuse (it is right isosceles with legs of length 1), and by Example 1.23(a) its area is exactly $\frac{1}{2}$. □

The statement scales: three closed squares of side $a > 0$, arbitrarily oriented, with pairwise disjoint interiors and non-obtuse centre triangle, span centre-triangle area at least $a^2/2$ (Exercise 1.6).

2. The angle hypothesis cannot be dropped

Example 1.37 (Obtuse degeneration)**.** Fix $0 < \varepsilon < \frac{1}{2}$ and take axis-parallel unit squares centred at $(0,0)$, $(1,\varepsilon)$, $(2,0)$ (Figure 1.4). Their shadows on the horizontal axis are the intervals $[-\frac{1}{2},\frac{1}{2}]$, $[\frac{1}{2},\frac{3}{2}]$, $[\frac{3}{2},\frac{5}{2}]$, whose interiors are pairwise disjoint. A common interior point $X = (x,y)$ of two of the squares would satisfy $|\langle X-P,(1,0)\rangle| < \frac{1}{2}$ for both centres P by Corollary 1.11, giving a point x in the interiors of two of these intervals; so no such X exists and the family is admissible.

By Example 1.23(b) the centre triangle has area ε, and by Example 1.30(b) it is obtuse at the middle vertex. Letting $\varepsilon \to 0$ shows that the infimum of the centre-triangle area over admissible triples, with no angle hypothesis, is 0.

3. The correct hypothesis-free bound

That bound is Proposition 1.35: Area $\geq \frac{1}{2}\sin\theta$ for every angle θ. The obtuse degeneration makes $\sin\theta$ small only by making an angle obtuse, so the two statements do not conflict.

4. There is no four-centre analogue

We need three definitions. Given a cyclic list q_0, q_1, q_2, q_3 of points (indices modulo 4), the *turn determinant* at q_k is $\det(q_k - q_{k-1},\, q_{k+1} - q_k)$: it measures how the closed polygon $q_0q_1q_2q_3q_0$ turns as it passes through q_k. The four points are in *strict convex position in that cyclic order* if all four turn determinants are strictly positive; the cyclic order matters, since the same four points in another order may give determinants of mixed signs. Finally, the *quadrilateral spanned* by such a list is the union of the two triangles $q_0q_1q_2$ and $q_0q_2q_3$; it is the convex hull of the four points, though the proof of that requires Chapter 4 and is not used below.

Proposition 1.38 (Four-centre infimum)**.** *Over all quadruples of unit squares with pairwise disjoint interiors whose centres are in strict convex position in some cyclic order, the infimum of the area of the quadrilateral they span is* 0*. This holds even when restricted to axis-parallel unit squares that are pairwise disjoint as closed sets, and the infimum is not attained.*

The construction is Exercise 1.7, and the full treatment is in Chapter 13. Theorem 1.1 is therefore specific to triples; there is no corresponding statement for larger families of centres.

Two remarks on terminology

- *Disjoint interiors, not disjoint.* Item 1 fails if one insists on disjoint closed squares: the extremal configuration touches along two edges and at a corner.

- *Contact.* No square is moved or rotated in the proof: the doubled area is minimised over the centres while all frames are held fixed, and from Chapter 5 onwards a branch is *active* when equality holds in a chosen frozen linear inequality, never when two squares are geometrically tangent. Later chapters say “complete contact” for “all three branches active”, in the same sense.

1.8 A map of the proof

The argument has the following shape.

> The proof assumes a counterexample, freezes one linear inequality per pair of squares, minimises the doubled area over the resulting compact set, converts first-order optimality at the minimiser into nonnegative multipliers, and finishes with
>
> $$2D \;=\; \lambda_1 H_1 + \lambda_2 H_2 + \lambda_3 H_3 \;\geq\; 2,$$
>
> so $D \geq 1$, contradicting $D \leq D_0 < 1$.

Layers 1–4 in the table below correspond to the four clauses of that sentence. Layer 0 supplies the language they all use, and the assembly is carried out in Chapter 13.

Layer	What it does	Built in	Used in
0. Language	orientation sign and the quarter turn J_σ; triangle geometry and the subcritical triangle lemma; scalar inequalities, concavity, half-angle coordinates; convex sets, support functions, Minkowski sums	Ch. 2, 3, 4	everywhere
1. Freeze	disjointness of a pair replaced by one linear inequality $\langle Q-P, n\rangle \geq H(F,G;n)$; ownership; the threshold $H(t) = (1+\cos t+\sin t)/2 \geq 1$; one owned branch per pair frozen once and for all	Ch. 4–5, applied in Ch. 9	Layers 2–4
2. Minimise	configuration $x = (P_2, P_3) \in \mathbb{R}^4$; feasible set closed and bounded; a minimiser exists, is acute, and has $0 < D \leq D_0 < 1$; all three frozen branches are active there	Ch. 6, 7 (with Ch. 2's subcritical lemma and Ch. 8's half-plane lemma), assembled in Ch. 9	Layer 3
3. Multipliers	no feasible descent direction at a minimiser; the finite Farkas lemma; multipliers $\lambda_i \geq 0$; Euler homogeneity gives $2D = \sum_i \lambda_i H_i$; the reciprocal force triangle and the sine system	Ch. 6, 7, 8, assembled in Ch. 10	Layer 4
4. Cases	oriented sines: some $s_i \leq 0$; all positive with cyclic owner tournament; all positive with transitive owner tournament — each forces $\sum_i \lambda_i H_i \geq 2$	Ch. 10 (nonpositive sine), Ch. 11 (cyclic), Ch. 12 (transitive), toolkit from Ch. 3	Ch. 13
Assembly	the three cases are exhaustive; the contradiction is drawn; sharpness	Ch. 13	—

What is used later is the statement of Theorem 1.1 and the vocabulary of §§1.2–1.5. The sharpness examples of §1.7 return in Chapter 13.

Exercises

Exercise 1.1 (Interiors). (a) Show that $\operatorname{int} X$ is the largest open subset of X: it is open, it is contained in X, and every open $U \subseteq X$ satisfies $U \subseteq \operatorname{int} X$. Deduce that $\operatorname{int}(\operatorname{int} X) = \operatorname{int} X$.

(b) Show that for all sets $X, Y \subseteq \mathbb{R}^n$ one has $\operatorname{int} X \cup \operatorname{int} Y \subseteq \operatorname{int}(X \cup Y)$ and $\operatorname{int}(X \cap Y) = \operatorname{int} X \cap \operatorname{int} Y$.

(c) Show that "$X \cap Y = \emptyset$" is strictly stronger than "$\operatorname{int} X \cap \operatorname{int} Y = \emptyset$", by exhibiting sets in $\mathbb{R}^2$ for which the second holds and the first fails. Then show that if two unit squares $Q(F,P)$ and $Q(G,R)$ have disjoint interiors, their intersection contains no interior point of either, i.e. $Q(F,P) \cap Q(G,R) \subseteq \big(Q(F,P) \setminus \operatorname{int} Q(F,P)\big) \cap \big(Q(G,R) \setminus \operatorname{int} Q(G,R)\big)$.

(d) For $k \geq 1$ let $U_k = \{x \in \mathbb{R} : x < 1/k\}$, which is open by Proposition 1.6(i). Show that $\bigcap_{k\geq 1} U_k$ is not open, and identify the step in the proof of Proposition 1.6(i) that fails for infinitely many indices.

Exercise 1.2 (Squares and frames). (a) Show that every closed square of side 1 in $\mathbb{R}^2$ is of the form $Q(F,P)$ for some frame F and some point P.

(b) Show that $Q(F',P') = Q(F,P)$ forces $P' = P$, and that exactly eight frames F' satisfy $Q(F',P) = Q(F,P)$.

(c) Deduce that the centre of a unit square is well defined, so that "the centres P_1, P_2, P_3" in Theorem 1.1 is unambiguous.

Exercise 1.3 (Determinants and area). (a) Show that $\mathrm{Area}(XYZ) \le \frac{1}{2}\|Y - X\|\,\|Z - X\|$ for all X, Y, Z, with equality if and only if $Y - X$ and $Z - X$ are orthogonal; that is, among triangles with two prescribed side lengths at a vertex, the right-angled one has the largest area.

(b) Deduce that Theorem 1.1 is already known in the right-angled case: if three unit squares have pairwise disjoint interiors and their centre triangle has a right angle, then its area is at least $\frac{1}{2}$. Which earlier result of this chapter supplies the two side lengths required, and why does the argument say nothing when the right angle is replaced by an acute one?

(c) Compute the three altitudes of the triangle with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$ from Corollary 1.19, and check that $\frac{1}{2}(\text{base}) \times (\text{height})$ gives the same answer for all three choices of base.

Exercise 1.4 (Non-obtuseness). (a) Let a triangle have side lengths $a \le b \le c$. Show that it is non-obtuse if and only if $c^2 \le a^2 + b^2$; that is, only the largest angle needs checking.

(b) Show that a triangle is non-obtuse if and only if at every vertex the inner product of the two edge vectors emanating from that vertex is nonnegative, and acute if and only if all three such inner products are strictly positive.

(c) Give an example of three pairwise distinct points that satisfy the non-obtuseness condition at two of the vertices but not at the third.

Exercise 1.5 (Separation for a single frame). Let $F = (e, f)$ be a frame and let $P, R \in \mathbb{R}^2$.

(a) Show that $\operatorname{int} Q(F, P)$ and $\operatorname{int} Q(F, R)$ are disjoint if and only if $\max\{|\langle R - P, e\rangle|, |\langle R - P, f\rangle|\} \ge 1$.

(b) Deduce that $\|R - P\| \ge 1$, recovering Lemma 1.32 in this case.

(c) Determine all such pairs with $\|R - P\| = 1$.

(d) Explain why the criterion in (a) is a disjunction of two linear conditions rather than a single one, and why this makes the set of admissible pairs nonconvex.

Exercise 1.6 (Sharpness of the all-angle bound, and scaling). (a) Consider the family of Example 1.37, with centres $P_1 = (0, 0)$, $P_2 = (1, \varepsilon)$ and $P_3 = (2, 0)$. Compute $\sin\theta$ exactly at the middle vertex, and show that the ratio of $\mathrm{Area}(P_1P_2P_3)$ to $\frac{1}{2}\sin\theta$ tends to 1 as $\varepsilon \to 0$. Deduce that the constant $\frac{1}{2}$ in Proposition 1.35 cannot be increased.

(b) Show that Theorem 1.1 implies the following: if three closed squares of side $a > 0$, arbitrarily oriented, have pairwise disjoint interiors and a non-obtuse centre triangle, then that triangle has area at least $a^2/2$.

Exercise 1.7 (No four-centre analogue). Fix $L > 1$ and $\varepsilon > 0$ and take axis-parallel unit squares centred at

$$p_0 = (0, 0), \qquad p_1 = (L, \varepsilon), \qquad p_2 = (2L, -\varepsilon), \qquad p_3 = (3L, 0).$$

(a) Show that the four closed squares are pairwise disjoint.

(b) Show that in the cyclic order p_0, p_2, p_3, p_1 all four turn determinants equal $3L\varepsilon > 0$, so that the centres are in strict convex position in that order, and that in the order p_0, p_1, p_2, p_3 the turn determinants are not all of one sign.

(c) Show that the quadrilateral spanned by p_0, p_2, p_3, p_1 is the union of the triangles $p_0p_2p_3$ and $p_0p_3p_1$, which meet only along the segment p_0p_3, and deduce from Definition 1.20 that its area is $3L\varepsilon$.

(d) Conclude that the infimum in Proposition 1.38 is 0, and explain why it is not attained.

Exercise 1.8 (A special case of Theorem 1.1)**.** Let S_1, S_2, S_3 be axis-parallel unit squares with pairwise disjoint interiors and centres $P_i = (x_i, y_i)$, and suppose that each pair is separated horizontally: $|x_i - x_j| \geq 1$ for all $i \neq j$. Assume the centre triangle is non-obtuse. Relabel so that $x_1 < x_2 < x_3$; this is permissible, since the hypotheses are symmetric in the three squares and relabelling changes the determinant only by a sign (Proposition 1.21(ii)). Put $u = P_1 - P_2$, $v = P_3 - P_2$. Prove, in the following steps, that $\operatorname{Area}(P_1P_2P_3) \geq 1$, twice the bound given by Theorem 1.1.

(a) Show that $u_1v_1 \leq -1$.

(b) Use non-obtuseness at P_2 to deduce that $u_2v_2 \geq 1$; in particular u_2 and v_2 are nonzero and of the same sign.

(c) Show that u_1v_2 and u_2v_1 have opposite signs, so that $|\det(u, v)| = |u_1v_2| + |u_2v_1|$.

(d) Apply the AM–GM inequality to conclude that $|\det(u, v)| \geq 2\sqrt{|u_1v_1|\,|u_2v_2|} \geq 2$.

Finally, explain which consequence of admissibility the horizontal-separation assumption strengthens (compare Remark 1.33(b), and the branches of Chapter 5), and why the argument fails once the three pairs are separated along three different directions.

Chapter 2

Orientation, the Quarter Turn, and Triangle Geometry

Assumed here.

- From Chapter 1. The determinant $\det(u, v) = u_1v_2 - u_2v_1$ and the area formula $\mathrm{Area}(XYZ) = \frac{1}{2}|\det(Y - X, Z - X)|$. The three equivalent forms of non-obtuseness at a vertex X (Proposition 1.25): (i) $\theta_X \le \pi/2$; (ii) $\langle Y - X, Z - X\rangle \ge 0$; (iii) the squared-side inequality $\|Y - Z\|^2 \le \|X - Y\|^2 + \|X - Z\|^2$. Lagrange's identity $\langle u, v\rangle^2 + \det(u, v)^2 = \|u\|^2\|v\|^2$ (Lemma 1.17) and its geometric reading $|\det(u, v)| = \|u\|\|v\| \sin\theta$ (Corollary 1.18). That three pairwise distinct collinear points are not non-obtuse, so that a non-obtuse triple has non-vanishing determinant (Lemma 1.28 and Corollary 1.29). The centre-separation lemma, that two admissible centres are at distance at least 1; this is the source of the hypothesis "all sides at least 1" in Section 2.8.
- From Part IA: the inner product $\langle u, v\rangle$ and norm $\|u\| = \sqrt{\langle u, u\rangle}$ on $\mathbb{R}^n$; the Cauchy–Schwarz inequality; orthonormal bases; 2×2 matrices as linear maps; orthogonal matrices; $\sin^2 + \cos^2 = 1$; the parametrisation of unit vectors of the plane as $(\cos\alpha, \sin\alpha)$.
- Not assumed and not used: topology and convexity. Several-variable calculus is used only in Proposition 2.6(ii) and the exercise attached to it, and only through the definition of differentiability, quoted in Section 2.2 from Chapter 6.

Sections 2.1–2.5 develop the sign of the determinant, the quarter turn J_σ (Section 2.3) with its identity $\langle J_\sigma u, v\rangle = \sigma \det(u, v)$, and the unit-vector forms of Cauchy–Schwarz. Sections 2.6–2.8 prove the subcritical triangle lemma.

2.1 The sign of a determinant: orientation of an ordered pair

The determinant of an ordered pair of plane vectors carries two pieces of information: a magnitude, used in Chapter 1 to compute areas, and a sign, which was discarded there. This section develops the sign.

Recall. The map $\det\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, $\det(u, v) = u_1v_2 - u_2v_1$, is bilinear and alternating; in particular

$$\det(v, u) = -\det(u, v), \quad \det(u, u) = 0, \quad \det(u, v + \lambda u) = \det(u, v) \quad (\lambda \in \mathbb{R}),$$

and $\det(u, v) = 0$ if and only if u, v are linearly dependent. The last displayed identity is *shear invariance*: sliding the second vector parallel to the first changes nothing. It is base-times-height in algebraic form; see Section 2.6.

Lemma 2.1 (Lagrange's identity in the plane; restated from Chapter 1)**.** *For all $u, v \in \mathbb{R}^2$,*

$$\langle u, v\rangle^2 + \det(u, v)^2 = \|u\|^2\, \|v\|^2.$$

This is Lemma 1.17, proved there by expanding both sides in coordinates. Two consequences, also from Chapter 1, are used repeatedly below. First, Cauchy–Schwarz in the plane: $\langle u, v\rangle^2 =$

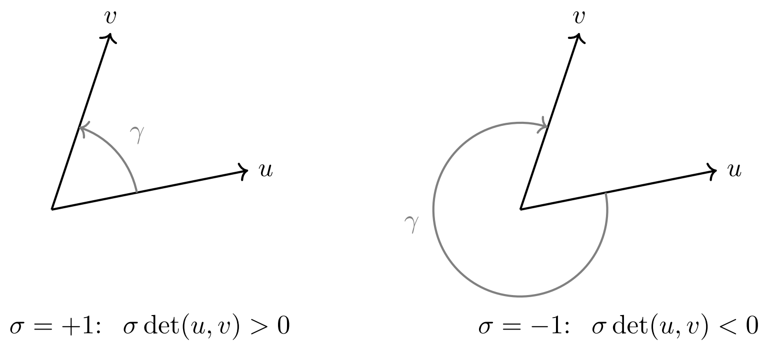


Figure 2.1: The same ordered pair (u, v), twice. The picture does not change; the sign σ decides in which sense the angle from u to v is swept, and hence the sign of $\sigma \det(u, v)$. On the left $\gamma \in (0, \pi)$, on the right $\gamma \in (\pi, 2\pi)$.

$\|u\|^2\|v\|^2 - \det(u, v)^2 \leq \|u\|^2\|v\|^2$, with equality if and only if $\det(u, v) = 0$, that is, if and only if u, v are linearly dependent. Second, if u, v are nonzero and $\theta \in [0, \pi]$ is the angle between them, defined as usual by $\cos\theta = \langle u, v\rangle/(\|u\|\|v\|)$, then by Corollary 1.18

$$|\det(u, v)| = \|u\| \, \|v\| \, \sin\theta. \tag{2.1}$$

Thus $|\det(u, v)|$ determines the angle between u and v but not the sense in which it is swept; that is what the sign records.

Definition 2.2 (orientation)**.** An ordered pair (u, v) of vectors of $\mathbb{R}^2$ is *positively oriented* if $\det(u, v) > 0$, *negatively oriented* if $\det(u, v) < 0$, and *degenerate* if $\det(u, v) = 0$. Given a sign $\sigma \in \{-1, +1\}$, the pair is *σ-positive* if $\sigma \det(u, v) > 0$. A triple (n_1, n_2, n_3) of nonzero vectors is in *positive cyclic order for σ* if all three of

$$\sigma \det(n_1, n_2), \qquad \sigma \det(n_2, n_3), \qquad \sigma \det(n_3, n_1)$$

are positive.

The third condition in the definition of positive cyclic order is not implied by the first two: for $\sigma = +1$ and the unit vectors at angles $0, \pi/6, \pi/3$, the first two determinants are $\sin(\pi/6) > 0$ but the third is $\sin(-\pi/3) < 0$. And positive cyclic order is invariant under a cyclic shift of (n_1, n_2, n_3), since a cyclic shift permutes the three displayed quantities among themselves, but fails after a transposition, since interchanging n_1 and n_2 replaces $\sigma \det(n_1, n_2)$ by its negative. (Chapter 10 takes "all three oriented sines are positive" as a standing hypothesis, and uses both observations when relabelling.)

Lemma 2.3 (directed angle)**.** *Fix $\sigma \in \{-1, +1\}$ and let u, v be unit vectors of $\mathbb{R}^2$. There is a unique $\gamma \in [0, 2\pi)$ with*

$$\langle u, v\rangle = \cos\gamma, \qquad \sigma \det(u, v) = \sin\gamma.$$

Moreover $\gamma \in (0, \pi)$ if and only if $\sigma \det(u, v) > 0$.

Proof. Write $u = (\cos\alpha, \sin\alpha)$ and $v = (\cos\beta, \sin\beta)$, which is possible since u, v are unit vectors. Then

$$\begin{aligned} \langle u, v\rangle &= \cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\beta - \alpha), \\ \det(u, v) &= \cos\alpha\sin\beta - \sin\alpha\cos\beta = \sin(\beta - \alpha). \end{aligned}$$

For $\sigma = +1$ let γ be the representative of $\beta - \alpha$ in $[0, 2\pi)$; for $\sigma = -1$ let γ be the representative of $\alpha - \beta$, using $\cos(-z) = \cos z$ and $\sin(-z) = -\sin z$. In both cases the two displayed equations hold. Uniqueness holds because $\gamma \mapsto (\cos\gamma, \sin\gamma)$ is injective on $[0, 2\pi)$. The final clause is the statement that $\sin\gamma > 0$ exactly for $\gamma \in (0, \pi)$. $\square$

We call the γ of Lemma 2.3 the *directed angle from u to v in the σ-orientation* and write $\gamma = \angle_\sigma(u, v)$ when the notation is needed. If $v \neq \pm u$ then $\angle_\sigma(v, u) = 2\pi - \angle_\sigma(u, v)$, since the two determinants are negatives of one another and the two inner products agree.

That three unit vectors in positive cyclic order have directed gaps $\angle_\sigma(n_i, n_{i+1})$ summing to 2π is a winding statement, proved in Chapter 11 where it is used.

2.2 Freezing σ: a signed objective in place of an absolute value

The quantity to be minimised is the area of a triangle, that is, $\frac{1}{2}|\det|$. An absolute value is not differentiable where its argument vanishes, and Chapters 6–10 need a polynomial objective. The remedy is to fix the sign in advance.

Example 2.4 (the absolute value is not differentiable)**.** Take $P_1 = (0, 0)$, $P_2 = (1, 0)$ and $P_3 = (0, \tau)$ with $\tau \in \mathbb{R}$. Then

$$\det(P_2 - P_1, P_3 - P_1) = \det\big((1, 0), (0, \tau)\big) = \tau,$$

so twice the area is $|\tau|$. This is not differentiable at $\tau = 0$: the one-sided derivatives there are $+1$ and -1. With a sign σ fixed in advance, $\tau \mapsto \sigma\tau$ is linear, hence infinitely differentiable.

The failure occurs at the collinear configuration $\tau = 0$, and the configurations studied below are never collinear (Lemma 2.18). What matters is rather that $x \mapsto |\det(x)|$ and $x \mapsto \sigma \det(x)$ are different functions, only the second of which is a polynomial; a proof that differentiates the objective, or that calls it quadratic, is a proof about the second.

Definition 2.5. Let P_1, P_2, P_3 be non-collinear points of $\mathbb{R}^2$. Their *orientation sign* is

$$\sigma = \operatorname{sign} \det(P_2 - P_1, P_3 - P_1) \in \{-1, +1\}.$$

Once a sign σ has been chosen, the *doubled signed area*, always called simply *the doubled area*, of an arbitrary triple (P_1, P_2, P_3) is

$$D = \sigma \det(P_2 - P_1, P_3 - P_1).$$

Throughout the book σ is a constant, computed once at a single configuration; only the points vary.

Differentiability. Part (ii) of the next proposition uses the definition of differentiability adopted in Chapter 6 (Definition 6.1): a function $f\colon \mathbb{R}^N \to \mathbb{R}$ is *differentiable at x* if there is a vector $g \in \mathbb{R}^N$ with

$$f(x + h) = f(x) + \langle g, h\rangle + R(h) \quad \text{for all } h, \qquad \frac{R(h)}{\|h\|} \to 0 \ \text{ as } h \to 0.$$

The only consequence needed reduces everything below to one-variable calculus: if f is differentiable at x then, for each fixed $v \in \mathbb{R}^N$, the one-variable function $t \mapsto f(x + tv)$ is differentiable at $t = 0$. Indeed $f(x+tv)-f(x)-t\langle g, v\rangle = R(tv)$ and, for $t \neq 0$ and $v \neq 0$, $|R(tv)/t| = \|v\|\cdot|R(tv)|/\|tv\| \to 0$. Contrapositively, to show that f is not differentiable at x it suffices to exhibit a single line through x along which the restriction of f fails to be differentiable. That is how part (ii) is proved.

Proposition 2.6 (freezing)**.** *Fix $\sigma \in \{-1, +1\}$ and write $D(P_1, P_2, P_3) = \sigma \det(P_2 - P_1, P_3 - P_1)$. Then:*

(i) $|D| = 2\operatorname{Area}(P_1P_2P_3)$, and $D \leq 2\operatorname{Area}(P_1P_2P_3)$ with equality if and only if $D \geq 0$;

(ii) D is a polynomial of degree 2 in the six coordinates of the three points, hence infinitely differentiable, whereas $2\operatorname{Area}$ fails to be differentiable at every collinear triple whose three points are not all equal;

(iii) D is unchanged if all three points are translated by a common vector;

(iv) D is unchanged by a cyclic shift of the labels, and changes sign under a transposition of two labels; explicitly

$$\det(P_2 - P_1, P_3 - P_1) = \det(P_3 - P_2, P_1 - P_2) = \det(P_1 - P_3, P_2 - P_3),$$

while interchanging P_2 and P_3 negates it;

(v) if S is any set of triples on which $D \geq 0$, then minimising D over S and minimising $2\,\mathrm{Area}$ *over S are the same problem: the same minimisers and the same minimum value.*

Proof. (i) By Chapter 1, $2\,\mathrm{Area} = |\det(P_2 - P_1, P_3 - P_1)| = |D|$ since $|\sigma| = 1$. Any real D satisfies $D \leq |D|$, with equality if and only if $D \geq 0$.

(ii) Writing $P_i = (x_i, y_i)$,

$$D = \sigma\big[(x_2 - x_1)(y_3 - y_1) - (y_2 - y_1)(x_3 - x_1)\big],$$

a polynomial each of whose monomials has degree 2; polynomials are differentiable in the sense recalled above, with gradient the vector of partial derivatives, and their partial derivatives are again polynomials, so D is infinitely differentiable. For the second claim, let (P_1, P_2, P_3) be collinear with not all three points equal, so $D = 0$ there, and suppose first that $P_2 \neq P_1$. Choose $w \in \mathbb{R}^2$ with $c := \det(P_2 - P_1, w) \neq 0$; this is possible because $\det(P_2 - P_1, \cdot)$ is a nonzero linear functional. Then for $t \in \mathbb{R}$,

$$D(P_1, P_2, P_3 + tw) = \det(P_2 - P_1, P_3 - P_1) + t \det(P_2 - P_1, w) = tc,$$

so the restriction of $2\,\mathrm{Area}$ to this line is $t \mapsto |tc|$, which has one-sided derivatives $\pm|c| \neq 0$ at $t = 0$ and is therefore not differentiable there; by the criterion just quoted, $2\,\mathrm{Area}$ is not differentiable at the triple. If instead $P_2 = P_1$, then $P_3 \neq P_1$, and the same argument applies with the roles of P_2 and P_3 exchanged, using $\det(\cdot, P_3 - P_1)$. (At a triple with $P_1 = P_2 = P_3 = P$ the function $2\,\mathrm{Area}$ is differentiable, with $g = 0$: writing $P_i = P + h_i$ and $h = (h_1, h_2, h_3)$ one has $2\,\mathrm{Area} = |\det(h_2 - h_1, h_3 - h_1)| \leq \|h_2 - h_1\|\,\|h_3 - h_1\| \leq 4\|h\|^2$, the first inequality by Lemma 2.1 and the second by the triangle inequality; so the remainder $R(h) = 2\,\mathrm{Area}$ satisfies $R(h)/\|h\| \to 0$.)

(iii) Each argument of det is a difference of two of the points, so a common translation cancels.

(iv) Put $a = P_2 - P_1$ and $b = P_3 - P_1$. Then $P_3 - P_2 = b - a$ and $P_1 - P_2 = -a$, so by bilinearity and $\det(a, a) = 0$,

$$\det(P_3 - P_2, P_1 - P_2) = \det(b - a, -a) = -\det(b, a) + \det(a, a) = \det(a, b).$$

Similarly $P_1 - P_3 = -b$ and $P_2 - P_3 = a - b$, so $\det(-b, a - b) = -\det(b, a) + \det(b, b) = \det(a, b)$. The transposition claim is antisymmetry: swapping P_2 and P_3 replaces $\det(a, b)$ by $\det(b, a) = -\det(a, b)$.

(v) On S the two functions are equal, by (i). ☐

Chapter 9 computes σ at an assumed counterexample and adds the constraint $D \geq 0$ to the minimisation, so that part (v) applies; parts (ii) and (iv) are used in Chapters 6 and 10, the latter to check that a relabelling leaves the displayed equations in form.

The constraint $D \geq 0$ cannot be dropped. On a set meeting both $\{D > 0\}$ and $\{D < 0\}$, minimising D is not the same as minimising area: a minimiser of D lies as far as possible inside $\{D < 0\}$, where the area is large. Only on a set contained in $\{D \geq 0\}$ do the two problems coincide (Exercise 2.6).

2.3 The quarter turn J_σ

Definition 2.7. For $\sigma \in \{-1, +1\}$ let $J_\sigma \colon \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation of the plane through the angle $\sigma\pi/2$, that is, the linear map with matrix

$$J_\sigma = \begin{pmatrix} 0 & -\sigma \\ \sigma & 0 \end{pmatrix}, \qquad \text{so} \qquad J_\sigma(x, y) = (-\sigma y,\ \sigma x).$$

Equivalently $J_\sigma = \sigma J_{+1}$, where $J_{+1}(x, y) = (-y, x)$.

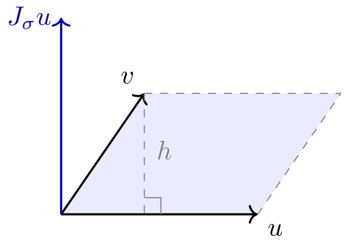


Figure 2.2: The identity $\langle J_\sigma u, v\rangle = \sigma \det(u, v)$ read as "base times signed height". The signed height of v above the line through u is $h = \sigma \det(u, v)/\|u\| = \langle J_\sigma u, v\rangle/\|u\|$; the shaded parallelogram has area $\|u\|\,|h| = |\det(u, v)|$. The configuration is drawn with $\sigma = +1$ and $\det(u, v) > 0$, so that $h > 0$ there and the dashed drop has length h.

We always write the operator as J_σ, carrying the sign explicitly, rather than speaking of rotation by ninety degrees.

Proposition 2.8 (properties of J_σ)**.** *Let $\sigma \in \{-1, +1\}$ and $u, v \in \mathbb{R}^2$. Then:*

(i) $J_\sigma^2 = -I$;
(ii) $\langle J_\sigma u, v\rangle = \sigma \det(u, v)$;
(iii) $\langle u, J_\sigma v\rangle = -\sigma \det(u, v) = \sigma \det(v, u)$; *equivalently* $J_\sigma^\mathsf{T} = -J_\sigma = J_{-\sigma}$;
(iv) $\langle J_\sigma u, u\rangle = 0$ *and* $\|J_\sigma u\| = \|u\|$;
(v) $\langle J_\sigma u, J_\sigma v\rangle = \langle u, v\rangle$ *and* $\det(J_\sigma u, J_\sigma v) = \det(u, v)$;
(vi) *if* $\|u\| = 1$ *and* $v = J_\sigma u$, *then* $\sigma \det(u, v) = 1$; *in particular* (u, v) *is* σ*-positive;*
(vii) *if* n *is a unit vector then* $(n, J_\sigma n)$ *is an orthonormal basis of* $\mathbb{R}^2$, *and it is* σ*-positively oriented.*

Proof. (i) $J_\sigma^2 = \sigma^2 J_{+1}^2 = J_{+1}^2$, and $J_{+1}^2(x, y) = J_{+1}(-y, x) = (-x, -y)$.

(ii) $\langle J_\sigma u, v\rangle = \langle(-\sigma u_2, \sigma u_1), (v_1, v_2)\rangle = \sigma(u_1 v_2 - u_2 v_1) = \sigma \det(u, v)$.

(iii) By symmetry of the inner product and (ii), $\langle u, J_\sigma v\rangle = \langle J_\sigma v, u\rangle = \sigma \det(v, u) = -\sigma \det(u, v)$. Comparing with (ii) gives $\langle J_\sigma^\mathsf{T} u, v\rangle = \langle u, J_\sigma v\rangle = -\langle J_\sigma u, v\rangle$ for all v, so $J_\sigma^\mathsf{T} = -J_\sigma$; and $-J_\sigma = J_{-\sigma}$ is immediate from the matrix.

(iv) Take $v = u$ in (ii) and use $\det(u, u) = 0$; and $\|J_\sigma u\|^2 = \sigma^2 u_2^2 + \sigma^2 u_1^2 = \|u\|^2$.

(v) $J_\sigma^\mathsf{T} J_\sigma = -J_\sigma^2 = I$ by (i) and (iii), so $\langle J_\sigma u, J_\sigma v\rangle = \langle J_\sigma^\mathsf{T} J_\sigma u, v\rangle = \langle u, v\rangle$. Since $\det J_\sigma = \sigma^2 = 1$, multiplicativity of the determinant gives $\det(J_\sigma u, J_\sigma v) = (\det J_\sigma)\det(u, v) = \det(u, v)$.

(vi) By (ii) with $v = J_\sigma u$ and (iv), $\sigma \det(u, J_\sigma u) = \langle J_\sigma u, J_\sigma u\rangle = \|u\|^2 = 1$.

(vii) By (iv), $J_\sigma n$ is a unit vector orthogonal to n, so $(n, J_\sigma n)$ is orthonormal; by (vi) it is σ-positive. □

Remark 2.9. The identity of Proposition 2.8(ii) characterises the operator: J_σ is the only linear map $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ with $\langle Lu, v\rangle = \sigma \det(u, v)$ for all u, v, since the right-hand side determines the linear functional $\langle Lu, \cdot\rangle$, hence determines Lu, for each u.

Corollary 2.10 (inversion and expansion)**.** *Let $\sigma \in \{-1, +1\}$.*

(a) *For* $d, f \in \mathbb{R}^2$: $J_\sigma d = f$ *if and only if* $d = -J_\sigma f$.
(b) *For a unit vector* n *and any* $v \in \mathbb{R}^2$,

$$v = \langle n, v\rangle\, n + \sigma \det(n, v)\, J_\sigma n.$$

Proof. (a) If $J_\sigma d = f$, apply J_σ and use $J_\sigma^2 = -I$: $-d = J_\sigma f$. Conversely if $d = -J_\sigma f$ then $J_\sigma d = -J_\sigma^2 f = f$.

(b) By Proposition 2.8(vii), $(n, J_\sigma n)$ is an orthonormal basis, so $v = \langle n, v\rangle n + \langle J_\sigma n, v\rangle J_\sigma n$, and $\langle J_\sigma n, v\rangle = \sigma \det(n, v)$ by part (ii) of the same proposition. □

Part (b) says that the coordinates of a vector in the frame $(n, J_\sigma n)$ are an inner product and a determinant. It is used for the "other-frame" width formula of Chapter 5 and for the basis expansion $n_2 = a n_1 + b n_3$ of Chapter 12. (Under the identification of $\mathbb{R}^2$ with $\mathbb{C}$, J_{+1} is multiplication by i; this is not used below.)

2.4 Using the identity in both directions

The identity $\langle J_\sigma u, v\rangle = \sigma \det(u, v)$ is used in both directions in later chapters. The three examples below illustrate each, and record how the sign σ propagates.

Example 2.11 (a numerical instance)**.** Let $\sigma = -1$, $u = (2, 1)$, $v = (1, 3)$. Then

$$\det(u, v) = 2 \cdot 3 - 1 \cdot 1 = 5, \qquad J_{-1}u = (-\sigma u_2, \sigma u_1) = (1, -2),$$

$$\langle J_{-1}u, v\rangle = 1 \cdot 1 + (-2) \cdot 3 = -5 = \sigma \det(u, v),$$

and $J_{-1}^2 u = J_{-1}(1, -2) = (-2, -1) = -u$, confirming $J_\sigma^2 = -I$ in this instance.

Note that $\det(u, v) = 5 > 0$, so the pair (u, v) is positively oriented, while $\sigma \det(u, v) = -5 < 0$, so it is not σ-positive. "Positively oriented" and "σ-positive" are different conditions; it is the second that is used below.

Example 2.12 (forwards: determinant to inner product)**.** Let $P_1, P_2, P_3 \in \mathbb{R}^2$ and put $d_1 = P_2 - P_1$. Then

$$D = \sigma \det(d_1, P_3 - P_1) = \langle J_\sigma d_1,\ P_3 - P_1\rangle.$$

With P_1 and P_2 held fixed, the right-hand side is a linear function of $P_3 - P_1$, whose defining vector $J_\sigma d_1$ has norm $\|d_1\|$ by Proposition 2.8(iv).

A bilinear expression with one slot held fixed is a linear functional in the other, and J_σ produces the vector representing it. (In Chapter 6 this vector is the gradient of the area.)

Example 2.13 (backwards: inner product to determinant, with an inversion)**.** Let n_1, n_2, n_3 be unit vectors, let $\lambda_2, \lambda_3 \geq 0$, put $f_i = \lambda_i n_i$, and suppose that some vector d satisfies

$$J_\sigma d = f_2 - f_3.$$

By Corollary 2.10(a), $d = -J_\sigma(f_2 - f_3) = J_\sigma(f_3 - f_2)$. Therefore, using Proposition 2.8(ii) and then bilinearity,

$$\begin{aligned}\langle d, n_1\rangle &= \langle J_\sigma(f_3 - f_2), n_1\rangle = \sigma \det(f_3, n_1) - \sigma \det(f_2, n_1)\\ &= \lambda_3\, \sigma \det(n_3, n_1) + \lambda_2\, \sigma \det(n_1, n_2),\end{aligned}$$

the last term by antisymmetry, $-\det(n_2, n_1) = \det(n_1, n_2)$.

Here one inverts with $J_\sigma^2 = -I$, takes an inner product, and reads each resulting inner product back as a determinant. This computation is the derivation of the sine system in Chapter 10, where the quantities $\sigma \det(n_i, n_{i+1})$ are named s_i.

Sign conventions. Three facts govern the appearances of σ.

(1) $\det(u, v)$ changes sign when its arguments are swapped; σ does not. So in $\sigma \det(u, v)$ only the arguments can be reordered, and reordering them introduces a sign.

(2) The quantities $\det(u, v)^2$, $|\det(u, v)|$, $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ are σ-free. Any final inequality expressed only in these must be σ-free too; if a σ survives into such a statement, there is an error.

(3) $J_{-\sigma} = -J_\sigma$, so replacing σ by $-\sigma$ negates every quantity built linearly from J_σ, and leaves unchanged every quantity built from the entries of list (2).

2.5 Cauchy–Schwarz in working form, and identities for unit vectors

The following is quoted repeatedly in later chapters.

Proposition 2.14 (unit-vector toolkit)**.** *(i)* (working form) *For every $d \in \mathbb{R}^2$ and every unit vector n,*

$$\langle d, n\rangle \ \leq\ |\langle d, n\rangle| \ \leq\ \|d\|,$$

with $\langle d, n\rangle = \|d\|$ if and only if $d = \|d\|\, n$.

(ii) For unit vectors n, m: $\langle n, m\rangle^2 + \det(n, m)^2 = 1$. Hence $|\langle n, m\rangle| \leq 1$ and $|\det(n, m)| \leq 1$, with $|\det(n, m)| = 1$ if and only if $n \perp m$, and $|\langle n, m\rangle| = 1$ if and only if $m = \pm n$.

(iii) If $a, b \in \mathbb{R}$ satisfy $a^2 + b^2 = 1$, then

$$1 \;\le\; |a| + |b| \;\le\; \sqrt{2},$$

with equality on the left if and only if $ab = 0$, and on the right if and only if $|a| = |b| = 1/\sqrt{2}$.

(iv) For unit vectors n, m: $1 \le |\langle n, m\rangle| + |\det(n, m)| \le \sqrt{2}$.

Proof. (i) Since $\|n\| = 1$,

$$0 \le \big\|d - \langle d, n\rangle n\big\|^2 = \|d\|^2 - 2\langle d, n\rangle^2 + \langle d, n\rangle^2 = \|d\|^2 - \langle d, n\rangle^2,$$

which gives $|\langle d, n\rangle| \le \|d\|$, and the first inequality is $t \le |t|$. Equality $|\langle d, n\rangle| = \|d\|$ forces $d = \langle d, n\rangle n$ by the same line; if moreover $\langle d, n\rangle = \|d\|$ then $d = \|d\|n$. The converse is a computation.

(ii) Lemma 2.1 with $\|n\| = \|m\| = 1$. Then $|\det(n, m)| = 1$ forces $\langle n, m\rangle = 0$, and conversely; and $|\langle n, m\rangle| = 1$ forces $\det(n, m) = 0$, that is, linear dependence, which for unit vectors means $m = \pm n$, and conversely.

(iii) $(|a| + |b|)^2 = a^2 + b^2 + 2|ab| = 1 + 2|ab| \ge 1$, with equality if and only if $ab = 0$. For the upper bound, $2|ab| \le a^2 + b^2 = 1$ by $(|a| - |b|)^2 \ge 0$, so $(|a| + |b|)^2 \le 2$, with equality if and only if $|a| = |b|$, which together with $a^2 + b^2 = 1$ means $|a| = |b| = 1/\sqrt{2}$.

(iv) Fix either sign σ and set $a = \langle n, m\rangle$ and $b = \langle J_\sigma n, m\rangle = \sigma \det(n, m)$. By Corollary 2.10(b) these are the coordinates of m in the orthonormal basis $(n, J_\sigma n)$, so $a^2 + b^2 = \|m\|^2 = 1$, and (iii) applies. Finally $|b| = |\det(n, m)|$, which does not depend on σ, in accordance with item (2) of the sign conventions above. □

Part (i) is used in Chapter 9 to convert each branch inequality $\langle d_i, n_i\rangle \ge H_i \ge 1$ into the side bound $\|d_i\| \ge 1$; part (ii) bounds the oriented sines by 1 in Chapter 10 and supplies $s_1^2 + s_2^2 = 1$ in Chapter 12; parts (iii) and (iv) give the two universal lower bounds of Chapter 5.

Example 2.15 (part (iv) in coordinates)**.** Take $n = (1, 0)$ and $m = (\cos\varphi, \sin\varphi)$. Then $\langle n, m\rangle = \cos\varphi$ and $\det(n, m) = \sin\varphi$, so

$$|\langle n, m\rangle| + |\det(n, m)| = |\cos\varphi| + |\sin\varphi|,$$

which equals 1 at the four axis directions $\varphi = 0,\ \pi/2,\ \pi,\ 3\pi/2$, and equals $\sqrt{2}$ at the four diagonal directions $\varphi = \pi/4,\ 3\pi/4,\ 5\pi/4,\ 7\pi/4$; both are the equality cases of Proposition 2.14(iii).

Write $t \in [0, \pi/4]$ for the distance from φ to the nearest multiple of $\pi/2$. Then $\{|\cos\varphi|, |\sin\varphi|\} = \{\cos t, \sin t\}$ and the sum above is $\cos t + \sin t$. Consequently

$$\frac{1 + \cos t + \sin t}{2} \;\ge\; 1 \qquad (0 \le t \le \pi/4),$$

since $(\cos t + \sin t)^2 = 1 + \sin 2t \ge 1$; this last computation is Proposition 2.14(iii) with $a = \cos t$ and $b = \sin t$. The displayed expression, and the reduction of φ to $t \in [0, \pi/4]$, recur in Chapter 5, where the relevant parameter ranges over an interval of length $\pi/4$ for this reason.

2.6 Altitudes, base times height, and $\frac{1}{2}ab\sin\theta$

Convention. For the rest of the chapter, a *triangle* XYZ is an ordered triple of pairwise distinct points of $\mathbb{R}^2$; it is *non-degenerate* if in addition the three points are not collinear. Thus every angle of a triangle is defined, in the sense of Chapter 1, and a degenerate triangle is one with all three points on a line. Statements that also hold when points coincide are stated in terms of the points rather than of a triangle.

Lemma 2.16 (rigid motions and reflections)**.** *Let $T(x) = Rx + b$ with R a 2×2 orthogonal matrix and $b \in \mathbb{R}^2$. Then T preserves distances and inner products of differences, and for all u, v, w,*

$$\det\big(Tu - Tv,\ Tw - Tv\big) = (\det R)\,\det(u - v,\ w - v).$$

In particular $|\det|$, unsigned areas, side lengths, angles and altitudes are unchanged, while the orientation sign is multiplied by $\det R = \pm 1$.

Proof. $Tu - Tv = R(u - v)$, and $\|Rz\|^2 = \langle R^\mathsf{T} Rz, z\rangle = \|z\|^2$, $\langle Rz, Rz'\rangle = \langle z, z'\rangle$, because $R^\mathsf{T} R = I$. The determinant identity is multiplicativity: $\det(Rz, Rz') = (\det R) \det(z, z')$. Angles are determined by inner products and norms, altitudes by distances. □

Every statement in Sections 2.7 and 2.8 concerns only lengths, angles and unsigned area, all of which are invariant under every map T of Lemma 2.16, reflections included; coordinates may therefore be chosen freely there. Any statement mentioning σ or D must instead be checked against $\det R$, since a reflection negates both.

Altitude and foot. Let X, Y, Z be points of $\mathbb{R}^2$ with $Y \neq Z$, and put $e = (Z - Y)/\|Z - Y\|$. The *foot of the altitude from* X is

$$\Pi = Y + \langle X - Y, e\rangle\, e,$$

the orthogonal projection of X onto the line through Y and Z; the *altitude from* X is the number $h_X = \|X - \Pi\|$.

Proposition 2.17 (metric formulas for a triangle). *Let $X, Y, Z \in \mathbb{R}^2$ with $Y \neq Z$, and let $\bar{D} = |\det(Y - X, Z - X)|$ be the doubled unsigned area.*

(i) $\bar{D}$ is unchanged by every permutation of X, Y, Z; in particular the same number $\bar{D}$ occurs in the formula below for each of the three vertices.

(ii) (base times height) $\|Z - Y\| \cdot h_X = \bar{D}$. *Consequently the three products (side)×(opposite altitude) are all equal.*

(iii) ($\frac{1}{2}ab\sin\theta$) *If $u = Y - X$ and $v = Z - X$ are nonzero, $a = \|u\|$, $b = \|v\|$, and $\theta \in [0, \pi]$ is the angle at X, then $\bar{D} = ab\sin\theta$, that is,* $\mathrm{Area}(XYZ) = \frac{1}{2}ab\sin\theta$.

Proof. (i) By Proposition 2.6(iv) the determinant is unchanged under cyclic shifts and negated under transpositions; the absolute value removes the remaining sign, and cyclic shifts together with one transposition generate all six permutations.

(ii) Put $L = \|Z - Y\|$, $e = (Z - Y)/L$ and $n = J_{+1}e$; by Proposition 2.8(vii), (e, n) is an orthonormal basis. Expanding $X - Y$ in it gives $X - Y = \langle X - Y, e\rangle e + \langle X - Y, n\rangle n$, so

$$X - \Pi = (X - Y) - \langle X - Y, e\rangle e = \langle X - Y, n\rangle\, n,$$

whence, using Proposition 2.8(ii) with $\sigma = +1$,

$$\begin{aligned} h_X &= |\langle X - Y, n\rangle| = |\langle J_{+1}e, X - Y\rangle| = |\det(e, X - Y)| \\ &= \frac{|\det(Z - Y, X - Y)|}{L} = \frac{\bar{D}}{L}, \end{aligned}$$

the last equality by (i). Multiplying by L gives the claim, and since $\bar{D}$ does not depend on the vertex, the three products are equal.

(iii) This is (2.1) applied to u, v, together with $\bar{D} = |\det(u, v)|$. □

Part (ii) is used in Chapter 9 to pass from an altitude bound to a side bound, and part (iii) gives the hypothesis-free bound $\mathrm{Area} \geq \frac{1}{2}\sin\theta$ of Chapter 13.

Non-obtuseness. Chapter 1 established that, at a vertex X of a triangle XYZ, the following are equivalent: the angle θ at X satisfies $\theta \leq \pi/2$; $\langle Y - X, Z - X\rangle \geq 0$; and $\|Y - Z\|^2 \leq \|Y - X\|^2 + \|Z - X\|^2$. A triangle is *non-obtuse* if this holds at all three vertices and *acute* (or *strictly acute*) if all three hold strictly. The inner-product form is used from here on, being the only one polynomial in the coordinates.

2.7 Where the foot of an altitude lies

Lemma 2.18 (non-obtuse triples are non-collinear; restated from Chapter 1). *Three pairwise distinct collinear points of $\mathbb{R}^2$ are not non-obtuse. Consequently, if P_1, P_2, P_3 are pairwise distinct and non-obtuse, then they are non-collinear and* $\det(P_2 - P_1, P_3 - P_1) \neq 0$.

Proof. The first assertion is Lemma 1.28 and the second is Corollary 1.29. □

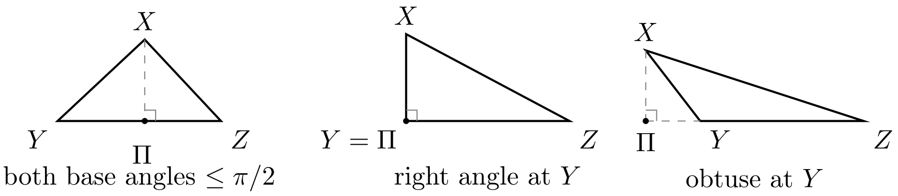


Figure 2.3: Lemma 2.19. The foot Π of the altitude from X lies inside the base when both base angles are non-obtuse (left), at Y when the angle there is right (centre), and outside the base when the angle at Y is obtuse (right), which is why the hypothesis is imposed.

In Chapter 9 non-obtuseness is imposed as a constraint, and this lemma is applied at every feasible configuration, not only at the assumed counterexample, to give non-degeneracy throughout the feasible set.

Lemma 2.19 (position of the foot)**.** *Let XYZ be a triangle. Put $L = \|Z - Y\|$, $e = (Z - Y)/L$ and $p = \langle X - Y, e\rangle$, so that the foot of the altitude from X is $\Pi = Y + pe$. Then*

$$\begin{aligned} p \ge 0 \quad &\Longleftrightarrow \quad \textit{the angle at } Y \textit{ is at most } \pi/2, \\ p \le L \quad &\Longleftrightarrow \quad \textit{the angle at } Z \textit{ is at most } \pi/2. \end{aligned}$$

In particular, if both base angles are non-obtuse then Π lies on the closed segment from Y to Z; moreover $p = 0$ exactly when the angle at Y is $\pi/2$, and $p = L$ exactly when the angle at Z is $\pi/2$.

Proof. Multiplying by $L > 0$,

$$pL = \langle X - Y, Z - Y\rangle,$$

and by the inner-product form of non-obtuseness the sign of the right-hand side is the sign of the cosine of the angle at Y; hence $p \ge 0$ if and only if that angle is at most $\pi/2$, with $p = 0$ exactly in the right-angled case. Similarly

$$(L - p)L = \langle Z - Y, Z - Y\rangle - \langle X - Y, Z - Y\rangle = \langle Z - X, Z - Y\rangle,$$

whose sign is the sign of the cosine of the angle at Z. The final assertion is the conjunction $0 \le p \le L$, which says that $\Pi = Y + pe$ lies between Y and Z. □

Corollary 2.20 (normal form for a non-obtuse base)**.** *Let XYZ be a non-degenerate triangle whose angles at Y and at Z are at most $\pi/2$. Then, after a map T as in Lemma 2.16, the configuration takes the form*

$$X = (0, h), \qquad Y = (-x, 0), \qquad Z = (y, 0),$$

with $h > 0$ the altitude from X, and $x, y \ge 0$ with $x + y = \|Z - Y\|$.

Proof. Keep the notation L, e, p, Π of Lemma 2.19. Translate Π to the origin and apply the rotation carrying e to $(1, 0)$; then $Y - \Pi = -pe$ becomes $(-p, 0)$ and $Z - \Pi = (L - p)e$ becomes $(L - p, 0)$, while X, being orthogonal to e relative to Π, becomes $(0, \pm h_X)$. If the sign is negative, reflect in the horizontal axis. Set $x = p$ and $y = L - p$; these are non-negative by Lemma 2.19, and $x + y = L$. Finally $h = h_X > 0$, since $h = 0$ would put X on the line through Y and Z, contradicting non-degeneracy. □

2.8 The subcritical triangle lemma

The hypotheses of the theorem below are those available in Chapter 9: sides at least 1 (from the centre-separation lemma of Chapter 1, or later from the frozen branch inequalities), non-obtuseness (imposed as a constraint), and doubled unsigned area less than 1 (the assumption that a counterexample exists).

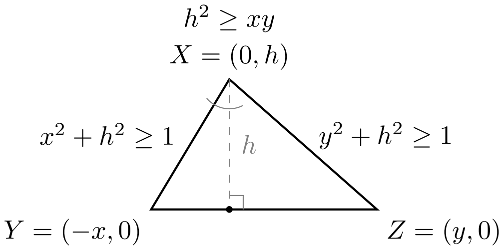


Figure 2.4: The normal form of Corollary 2.20 with the three inequalities used in the proof of Theorem 2.21: the two adjacent sides are at least 1, and non-obtuseness at X reads $h^2 \geq xy$.

Notation. No orientation is frozen in Sections 2.6–2.8, so the objective throughout them is the unsigned quantity $\bar{D} = 2\,\mathrm{Area}$, called the *doubled unsigned area*. The unbarred symbol D retains its meaning, the doubled signed area $\sigma \det(P_2 - P_1, P_3 - P_1)$, and "the doubled area" without qualification refers to that signed quantity. On the configurations of Chapter 9, where $D \geq 0$, the two agree.

Theorem 2.21 (subcritical triangles)**.** *Let XYZ be a non-obtuse triangle all of whose sides have length at least* 1*. Then:*

(i) every altitude of XYZ has length at least $1/\sqrt{2}$;

(ii) if, in addition, the doubled unsigned area $\bar{D} = 2\,\mathrm{Area}(XYZ)$ satisfies $\bar{D} < 1$, then every side has length strictly less than $\sqrt{2}$, and the triangle is strictly acute.

Proof. The three vertices are distinct, since all sides have length at least 1, and they are non-collinear by Lemma 2.18; so the triangle is non-degenerate and $\bar{D} > 0$.

(i) Fix a vertex, and relabel the triangle so that it is X, the other two being Y and Z; this is permissible because both hypotheses, non-obtuseness and "all sides at least 1", and the conclusion of (i) are symmetric in the three vertices, so the data keep their form under any relabelling. The base angles, at Y and at Z, are at most $\pi/2$ by hypothesis, so Corollary 2.20 applies; since the assertion to be proved concerns only lengths and angles, Lemma 2.16 permits us to assume that

$$X = (0, h), \quad Y = (-x, 0), \quad Z = (y, 0), \qquad h > 0,\ x \geq 0,\ y \geq 0,$$

with h the altitude from X (Figure 2.4). The two sides adjacent to X have length at least 1, which reads

$$x^2 + h^2 \geq 1, \qquad y^2 + h^2 \geq 1. \tag{2.2}$$

Non-obtuseness at X reads

$$0 \leq \langle Y - X, Z - X\rangle = \big\langle(-x, -h), (y, -h)\big\rangle = -xy + h^2, \qquad \text{that is,} \qquad h^2 \geq xy.$$

Suppose, for contradiction, that $h^2 < \frac{1}{2}$. Then by (2.2),

$$x^2 \geq 1 - h^2 > \tfrac{1}{2} > h^2,$$

and since $x \geq 0$ and $h > 0$ this forces $x > h$; the same argument gives $y > h$. In particular $y > h > 0$, so

$$xy > hy \geq h \cdot h = h^2,$$

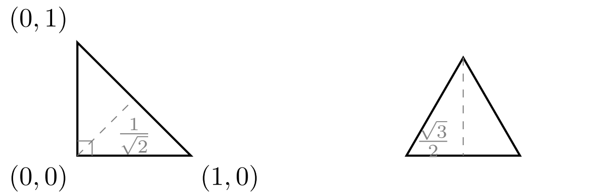
contradicting $h^2 \geq xy$. Hence $h^2 \geq \frac{1}{2}$, that is, $h \geq 1/\sqrt{2}$. The vertex was arbitrary, so all three altitudes are at least $1/\sqrt{2}$.

(ii) Assume $\bar{D} < 1$. By Proposition 2.17(ii), each side equals $\bar{D}$ divided by the altitude to it, so by part (i) each side has length at most $\bar{D}\sqrt{2} < \sqrt{2}$.

For strict acuteness, suppose some angle were exactly $\pi/2$. The two sides adjacent to it have lengths $a, b \geq 1$, so by Proposition 2.17(iii),

$$\bar{D} = ab \sin(\pi/2) = ab \geq 1,$$

contradicting $\bar{D} < 1$. Since the triangle is non-obtuse, no angle exceeds $\pi/2$ either; hence every angle is strictly less than $\pi/2$. $\square$



$\bar{D} = 1$, sides $1, 1, \sqrt{2}$ $\qquad$ $\bar{D} = \sqrt{3}/2$, sides $1, 1, 1$

Figure 2.5: Left: the critical triangle, on which the altitude bound of Theorem 2.21(i) is attained; its altitudes are $1, 1$ and $1/\sqrt{2}$. Right: the unit equilateral triangle, for which $\bar{D} < 1$.

Remark 2.22 (the critical value, and sharpness). *The critical value is* $\bar{D} = 1$*.* The right isosceles triangle with vertices $(0, 0), (1, 0), (0, 1)$ has sides $1, 1, \sqrt{2}$, is non-obtuse (its angles are $\pi/2, \pi/4, \pi/4$), and has $\bar{D} = 1$. Its altitudes are $1, 1$ and $1/\sqrt{2}$, so it satisfies the hypotheses of part (i) and attains the constant $1/\sqrt{2}$ there, which therefore cannot be raised. The constant $\sqrt{2}$ in part (ii) is sharp only in a weaker sense: no triangle satisfying the hypothesis of (ii) attains it, since the bound there is strict and the critical triangle, whose longest side is $\sqrt{2}$, has $\bar{D} = 1$ and so is excluded; but $\sqrt{2}$ is the supremum, because the isosceles triangle with two sides of length 1 and apex angle θ has third side $2\sin(\theta/2)$ and $\bar{D} = \sin\theta$, and for $\pi/3 \le \theta < \pi/2$ it has all sides at least 1, is non-obtuse (its base angles are $(\pi - \theta)/2 < \pi/2$) and satisfies $\bar{D} < 1$; letting $\theta \uparrow \pi/2$ drives $\bar{D} \uparrow 1$ and the longest side $2\sin(\theta/2) \uparrow \sqrt{2}$. The critical triangle is not strictly acute, which shows that the hypothesis $\bar{D} < 1$ in part (ii) cannot be weakened to $\bar{D} \le 1$. (This triangle is the equality case of Theorem 1.1: three axis-parallel unit squares centred at three vertices of a unit square.)

The subcritical regime is not empty. The equilateral triangle of side 1 is non-obtuse, has all sides at least 1, and has $\bar{D} = \sqrt{3}/2 < 1$; all its altitudes equal $\sqrt{3}/2 \ge 1/\sqrt{2}$ and all its sides equal $1 < \sqrt{2}$, as Theorem 2.21 requires. So the theorem is not vacuous. It also shows that triangle geometry alone cannot give $\bar{D} \ge 1$: closing the gap between $\sqrt{3}/2$ and 1 requires the square-packing hypotheses of Theorem 1.1. Exercise 2.7 shows that $\sqrt{3}/2$ is the exact bound obtainable from triangle geometry alone.

Non-obtuseness is necessary in part (i). For $0 < \varepsilon < 1$, the triangle with vertices $(0, 0), (2, 0), (1, \varepsilon)$ has sides $2, \sqrt{1 + \varepsilon^2}, \sqrt{1 + \varepsilon^2}$, all at least 1, but its altitude from $(1, \varepsilon)$ equals ε, which tends to 0. It is obtuse at $(1, \varepsilon)$, since $\langle(-1, -\varepsilon), (1, -\varepsilon)\rangle = \varepsilon^2 - 1 < 0$. This is the family of Example 1.30(b), with the vertices in a different order. It is used in Example 1.37 to show that the centre-triangle area has no positive lower bound without an angle hypothesis, and again in the same role in Chapter 13.

In Chapter 9, at every configuration satisfying the frozen branch inequalities, Proposition 2.14(i) gives all three sides length at least 1, non-obtuseness is imposed as a constraint, and the assumed counterexample gives $D \le D_0 < 1$. Theorem 2.21 then confines the feasible configurations to a bounded region, which yields compactness, and makes the minimiser strictly acute, so that Chapter 7 may treat acuteness as an open condition persisting under small motions.

Exercises

Exercise 2.1. (a) With $u = (3, 1)$, $v = (1, 2)$ and $\sigma = -1$: compute $\det(u, v)$ and $J_\sigma u$, verify both $\langle J_\sigma u, v\rangle = \sigma \det(u, v)$ and $J_\sigma^2 u = -u$, and say whether (u, v) is positively oriented and whether it is σ-positive.

(b) For arbitrary $u, v \in \mathbb{R}^2$ and $\sigma \in \{-1, +1\}$, show that if (u, v) is σ-positive then (v, u) is $(-\sigma)$-positive.

(c) For arbitrary $u, v \in \mathbb{R}^2$, $\sigma \in \{-1, +1\}$ and $\lambda \in \mathbb{R}$, show that $\sigma \det(u, v + \lambda u) = \sigma \det(u, v)$, and explain why this is the statement that shearing a triangle along its base preserves its area.

Exercise 2.2. Prove directly from Definition 2.7, without quoting Proposition 2.8:

(a) $\langle u, J_\sigma v\rangle = -\langle J_\sigma u, v\rangle$ for all $u, v \in \mathbb{R}^2$;

(b) $\det(J_\sigma u, J_\sigma v) = \det(u, v)$ for all $u, v \in \mathbb{R}^2$;
(c) J_σ is the only linear map $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ satisfying $\langle Lu, v\rangle = \sigma \det(u, v)$ for all $u, v \in \mathbb{R}^2$.

Exercise 2.3. Let n, m be unit vectors of $\mathbb{R}^2$.
(a) Prove $\langle n, m\rangle^2 + \det(n, m)^2 = 1$, and deduce $1 \le |\langle n, m\rangle| + |\det(n, m)| \le \sqrt{2}$, identifying both equality cases.
(b) Show that there is a unique $t \in [0, \pi/4]$ with $\{|\langle n, m\rangle|, |\det(n, m)|\} = \{\cos t, \sin t\}$, and that then $|\langle n, m\rangle| + |\det(n, m)| = \cos t + \sin t$.
(c) Show that $1 \le \cos t + \sin t \le \sqrt{2}$ for $t \in [0, \pi/4]$, with equality on the left only at $t = 0$ and on the right only at $t = \pi/4$. Deduce that $(1 + \cos t + \sin t)/2 \ge 1$ on $[0, \pi/4]$, with equality if and only if $t = 0$.

Exercise 2.4. Let $\sigma \in \{-1, +1\}$ and let n be a unit vector.
(a) Prove that every $v \in \mathbb{R}^2$ satisfies $v = \langle n, v\rangle\, n + \sigma \det(n, v)\, J_\sigma n$.
(b) Deduce that if u, v are unit vectors with $\sigma \det(u, v) = 0$ then $v = \pm u$, and that if $\sigma \det(u, v) = 1$ then $v = J_\sigma u$.
(c) Let n_1, n_3 be orthogonal unit vectors with $\sigma \det(n_3, n_1) = 1$, and let n_2 be any unit vector. Writing $s_1 = \sigma \det(n_1, n_2)$ and $s_2 = \sigma \det(n_2, n_3)$, prove that $s_1^2 + s_2^2 = 1$.

Exercise 2.5. Let XYZ be a triangle, and keep the notation L, e, p, Π of Lemma 2.19.
(a) Show that Π lies in the open segment from Y to Z if and only if both base angles are strictly less than $\pi/2$.
(b) Deduce that if the triangle is obtuse at X, then the foot of the altitude from X lies in the open segment YZ, while the feet of the other two altitudes lie strictly outside their bases. (Which side of the base does each lie on?)
(c) Show that a triangle all of whose sides have length at least 1 and all of whose altitudes have length at least $1/\sqrt{2}$ need not be non-obtuse. Conclude that part (i) of Theorem 2.21 is an implication and not an equivalence.

Exercise 2.6. Fix $\sigma = +1$ and, normalising $P_1 = 0$, let $D(P_2, P_3) = \det(P_2, P_3)$ on $\mathbb{R}^2 \times \mathbb{R}^2$.
(a) Show that D is a polynomial, while $(P_2, P_3) \mapsto 2\,\mathrm{Area}(0, P_2, P_3)$ fails to be differentiable at every point at which P_2 and P_3 are linearly dependent but not both zero. (Use the definition of differentiability quoted in Section 2.2, and the criterion recorded there: it is enough to restrict to a suitable line. Show also that at $P_2 = P_3 = 0$ the function is differentiable, so that the exclusion is necessary.)
(b) Show that the two functions agree on $\{D \ge 0\}$, and that minimising either over a subset of $\{D \ge 0\}$ gives the same minimisers and the same value.
(c) Exhibit a closed bounded set $S \subseteq \mathbb{R}^2 \times \mathbb{R}^2$ on which the minimum of D is strictly negative while the minimum of $2\,\mathrm{Area}$ is 0, and say which hypothesis of Proposition 2.6(v) fails.

Exercise 2.7. Let XYZ be a non-obtuse triangle with all sides of length at least 1, and let $\bar{D}$ be its doubled unsigned area.
(a) Using Proposition 2.17(iii), show that $\sin\theta \le \bar{D}$ for every angle θ of the triangle.
(b) Deduce that every angle satisfies $\theta \le \arcsin\bigl(\min\{\bar{D}, 1\}\bigr)$, and, using the fact that the three angles sum to π, that $\bar{D} \ge \sqrt{3}/2$.
(c) Show that this bound is attained, and explain why it shows that Theorem 1.1 cannot be proved from triangle geometry alone.

Chapter 3

Inequalities, Concavity, and Half-Angle Coordinates

Assumed here.

- From earlier chapters: nothing beyond the notational conventions of Chapter 1. Two formulas are quoted from later chapters, without their geometric meaning: the threshold function $H(t) = (1 + \cos t + \sin t)/2$ of Chapter 5 and the constraint $uv + vw + wu = 1$ of Chapter 11.
- From Part IA: the mean value theorem; the Cauchy–Schwarz inequality in $\mathbb{R}^n$; the addition formulas for sin and cos; monotone functions and their inverses; arccot as the inverse of cot on $(0, \pi)$.
- Used later in: Chapters 11 and 12, which quote the results below by number.
- Notation. Convex-combination weights are written θ, since λ is reserved for multipliers; the elementary symmetric polynomials of §3.6 are written e_1, e_2, e_3.

The geometry, optimisation and duality of the preceding and following chapters reduce to two explicit inequalities between real numbers. This chapter collects the techniques those two inequalities require, with no geometry attached. Three of them are new to a Part IA reader: the Engel form of Cauchy–Schwarz, the use of concavity to obtain subadditivity, and the half-angle substitution used as a coordinate system.

3.1 The two targets

The following two statements are proved in Chapters 11 and 12. They are recorded here as the targets of the techniques below.

> **The cyclic target** (proved in Chapter 11). Let $u, v, w > 0$ satisfy $uv + vw + wu = 1$, and set
> $$\Phi(q) = \frac{(1+q)\max\{1, q\}}{1+q^2} \qquad (q > 0).$$
> Then
> $$\Phi(u)\,(u+w) + \Phi(v)\,(u+v) + \Phi(w)\,(v+w) \;\geq\; 4.$$
>
> **The transitive target** (proved in Chapter 12). For $t \in (0, \pi/2)$ put $r = \tan(t/2) \in (0, 1)$. Then the three quantities
> $$\alpha = \frac{r}{(1+r)(1+r^2)}, \qquad \beta = \frac{1+r+r^2}{2(1+r)},$$
> $$K = \frac{r(1-r)\big(3 - 2r + 2r^2 - r^3\big)}{2(1+r^2)^2}$$
> are nonnegative, and (α, β) is the unique solution of an explicit 2×2 linear system.

The pairing in the cyclic target is rigid: $\Phi(u)$ carries the factor $u + w$, not $u + v$. Chapter 11 explains where the pairing comes from; here it is part of the statement.[1]

The table lists each technique with the section that develops it and the place it is used.

Technique	§	Used at
$u + 1/u \geq 2$	3.2	final step of Ch. 11, Case 2
cross-multiplication, sign rules	3.3	Ch. 11, Case 2 (middle term)
subtract, common denominator, factor	3.4	all of Ch. 11 Case 2; Ch. 12
polynomial bounds on $[0, 1]$	3.5	$K \geq 0$ in Ch. 12
symmetric functions of three variables	3.6	Ch. 11, Case 1 denominator
Engel form of Cauchy–Schwarz	3.7	Ch. 11, Case 1
concavity, chord form	3.8	Ch. 12, frame excess
subadditivity	3.9	Ch. 12, frame excess
extension past a critical point	3.10	Ch. 12, frame excess
half-angle coordinates r, q	3.11	Chs. 11, 12 throughout
cotangent addition and supplement	3.12	Ch. 11, the constraint $uv + vw + wu = 1$

3.2 AM–GM and $u + 1/u \geq 2$

Lemma 3.1 (two-variable AM–GM)**.** *For $a, b \geq 0$ we have $\frac{1}{2}(a + b) \geq \sqrt{ab}$, with equality if and only if $a = b$.*

Proof. $a + b - 2\sqrt{ab} = (\sqrt{a} - \sqrt{b})^2 \geq 0$, and this vanishes exactly when $\sqrt{a} = \sqrt{b}$, that is, exactly when $a = b$. □

Corollary 3.2. *For $u > 0$ we have $u + \dfrac{1}{u} \geq 2$, with equality if and only if $u = 1$.*

Proof. (i) Apply Lemma 3.1 with $a = u$ and $b = 1/u$: then $\sqrt{ab} = 1$, so $u + 1/u \geq 2$, with equality exactly when $u = 1/u$, that is, when $u = 1$.

(ii) Directly,

$$u + \frac{1}{u} - 2 = \frac{u^2 - 2u + 1}{u} = \frac{(u - 1)^2}{u},$$

a square divided by a positive number, hence nonnegative, and zero exactly when $u = 1$. □

Remark 3.3 (visibly nonnegative). Proof (ii) is the pattern followed throughout this chapter. To prove $X \geq Y$, exhibit the difference $X - Y$ as a single fraction whose denominator has been checked positive and whose numerator is a sum of products of factors whose signs are immediate from the standing hypotheses.

We call a quantity *visibly nonnegative* when it is presented as a sum of products of factors each of which is nonnegative by inspection given the hypotheses in force. Such an expression requires no case analysis and no numerical estimate: the factors are checked one at a time. Proof (ii) also yields the defect $(u - 1)^2/u$, and with it the equality case; proof (i) yields only the inequality.

Example 3.4. For $u, v > 0$, $(u + v)\left(\frac{1}{u} + \frac{1}{v}\right) \geq 4$.

By Lemma 3.1 twice, $u + v \geq 2\sqrt{uv}$ and $\frac{1}{u} + \frac{1}{v} \geq 2/\sqrt{uv}$; multiplying the two (both sides positive) gives 4. Alternatively, expanding and subtracting,

$$(u + v)\left(\frac{1}{u} + \frac{1}{v}\right) - 4 = 2 + \frac{u}{v} + \frac{v}{u} - 4 = \frac{u^2 + v^2 - 2uv}{uv} = \frac{(u - v)^2}{uv} \;\geq\; 0,$$

with equality exactly when $u = v$. The second computation follows Remark 3.3 and yields the equality case.

No other inequality between means is used in this book. Corollary 3.2 is used once, as the last line of one case of Chapter 11.

[1] The paper on which this book is based writes $F(q)$ for the function we call $\Phi(q)$, and G for the function we call E in §3.8. Both are renamed here because F and G denote frames throughout the book.

3.3 When cross-multiplication is legitimate

Chapter 11 cross-multiplies at a point where the required positivity follows from the hypotheses rather than being evident from the expression. The rules are therefore recorded here.

Lemma 3.5 (sign rules). *Let $m > 0$ and let x, k be real. Then $x/m \geq k$ if and only if $x \geq km$. If a, c are real and $b, d > 0$, then $a/b \geq c/d$ if and only if $ad \geq bc$. Finally, if $0 < \eta < \xi$ and $\kappa > 0$, then $\kappa/\xi < \kappa/\eta$.*

Proof. Multiplication by a positive number preserves the order relation and multiplication by a negative number reverses it; division by a positive number is multiplication by its (positive) reciprocal. For the first claim multiply $x/m \geq k$ by $m > 0$, and multiply $x \geq km$ by $1/m > 0$. For the second multiply $a/b \geq c/d$ by $bd > 0$, and conversely multiply $ad \geq bc$ by $1/(bd) > 0$. For the third, $\kappa/\eta - \kappa/\xi = \kappa(\xi - \eta)/(\eta\xi) > 0$ since $\kappa > 0$, $\xi - \eta > 0$ and $\eta\xi > 0$. □

Example 3.6 (a computation from Chapter 11). Let $u > 1$ and $0 < v, w < 1$ satisfy $uv+vw+wu = 1$. We show

$$\frac{1+v}{v+w} \;\geq\; \frac{(1+v)^2}{1+v^2}.$$

Rewrite the constraint by collecting the two terms containing u:

$$u(v+w) + vw = 1.$$

Now use this to eliminate the constant 1 from the difference of the two denominators:

$$\begin{aligned} 1 + v^2 - (v+w)(1+v) &= u(v+w) + vw + v^2 - (v+w)(1+v) \\ &= (v+w)\big(u + v - (1+v)\big) \\ &= (u-1)(v+w) \;>\; 0, \end{aligned}$$

where in the middle line we used $vw + v^2 = v(v+w)$, and the final strict inequality holds because $u > 1$ and $v, w > 0$. Hence

$$1 + v^2 \;>\; (v+w)(1+v).$$

Before dividing, record the signs: $1 + v > 0$, $v + w > 0$, $1 + v^2 > 0$ and $(1 + v)^2 > 0$. Apply the third clause of Lemma 3.5 with $\kappa = 1$, $\eta = (v+w)(1+v)$ and $\xi = 1+v^2$, legitimate since $0 < \eta < \xi$ by the strict inequality just proved:

$$\frac{1}{(v+w)(1+v)} \;>\; \frac{1}{1+v^2}, \qquad \text{hence} \qquad \frac{1}{v+w} \;>\; \frac{1+v}{1+v^2}$$

after multiplying by $1 + v > 0$; multiplying once more by $1 + v > 0$ gives the claim, indeed with strict inequality.

Both moves recur below: the signs are checked before any division, and the constraint is used to eliminate the constant 1 from a difference of denominators.

Remark 3.7 (the sign hypotheses are necessary). From $-1 \leq 1$ one may not conclude $-1/x \leq 1/x$: for $x = -1$ the two sides are 1 and -1. From $a/b \geq c/d$ one may not conclude $ad \geq bc$ without knowing the signs of b and d: with $a = 1$, $b = -1$, $c = -2$, $d = 1$ we have $a/b = -1 \geq -2 = c/d$, but $ad = 1 < 2 = bc$. The cross-multiplications below therefore state the positivity of both denominators at the point of use.

3.4 Rational inequalities: subtract, common denominator, factor

Remark 3.8 (Method). The standard recipe, in five steps.

1. Move everything to one side: reduce $X \geq Y$ to $X - Y \geq 0$.
2. Put $X - Y$ over a single denominator.
3. Check the sign of that denominator from the hypotheses, in writing.

4. Factor the numerator into visibly nonnegative pieces.
5. Read the equality case off the factorisation.

Step 4 admits no general recipe. The usual tactic is to peel off one estimate at a time rather than expanding everything at once: replace X by a simpler lower bound with a visibly nonnegative error, then repeat. Full expansion is preferable when the expression is small. Both are illustrated below.

Example 3.9 (a full expansion)**.** For $v \in [0,1)$,

$$\frac{(1+v)^2}{1+v^2} + \frac{1-v}{1+v} - 2 \;=\; \frac{2v^2(1-v)}{(1+v^2)(1+v)} \;\geq\; 0.$$

Indeed, the common denominator is $(1+v^2)(1+v)$, which is positive because $v \geq 0$; over it the numerator is

$$\begin{aligned}&(1+v)^3 + (1-v)(1+v^2) - 2(1+v^2)(1+v)\\ &\quad = (1+3v+3v^2+v^3) + (1-v+v^2-v^3) - (2+2v+2v^2+2v^3)\\ &\quad = 2v^2 - 2v^3 = 2v^2(1-v).\end{aligned}$$

Here $v < 1$ gives $1-v > 0$ and $v \geq 0$ gives $1+v > 0$; the other denominator, $1+v^2$, is positive for every real v. Equality holds when $v = 0$.

Example 3.10 (peeling: two estimates from Chapter 11)**.** Let $u > 1$ and $0 < v, w < 1$.

(a) We claim

$$\frac{u(1+u)}{u+v} - u - \frac{1-v}{1+v} \;=\; \frac{v(1-v)(u-1)}{(u+v)(1+v)} \;\geq\; 0.$$

Peel in two stages. First,

$$\frac{u(1+u)}{u+v} - u = \frac{u(1+u) - u(u+v)}{u+v} = \frac{u(1-v)}{u+v},$$

which already exhibits the first term as u plus something positive. Second,

$$\frac{u(1-v)}{u+v} - \frac{1-v}{1+v} = (1-v)\cdot\frac{u(1+v)-(u+v)}{(u+v)(1+v)} = (1-v)\cdot\frac{v(u-1)}{(u+v)(1+v)}.$$

The denominator $(u+v)(1+v)$ is positive since $u, v > 0$; the numerator is the product of $1-v > 0$, $v > 0$ and $u - 1 > 0$.

(b) Likewise

$$\frac{1+w}{w+u} - \frac{1}{u} = \frac{u(1+w)-(w+u)}{u(w+u)} = \frac{w(u-1)}{u(w+u)} \;>\; 0,$$

a direct computation: the denominator is positive since $u, w > 0$, and the numerator is a product of $w > 0$ and $u - 1 > 0$. In particular the estimate in (b) is never tight, both factors being strictly positive.

3.5 Bounding a low-degree polynomial on an interval

The polynomial bounds needed later are obtained by one of three techniques, illustrated on a cubic below.

(a) *Regrouping into nonnegative pieces.* On $[0,1]$ the quantities r, $1-r$, r^k and $2-r$ are all nonnegative; a polynomial written as a sum of products of such factors is visibly nonnegative there, and the constant summand is then a lower bound for it on the interval, though not in general its minimum.

(b) *Monotonicity.* Differentiate, show the derivative has constant sign on the interval, and evaluate at the appropriate endpoint. This is the route to take when no regrouping is apparent.

(c) *Term-by-term estimation.* For $r \in [0,1]$ and $k \geq j \geq 0$ we have $r^k \leq r^j$, so a monomial of high degree may be discarded upwards or downwards against one of lower degree. This is crude but often sufficient.

Example 3.11 (the Chapter 12 cubic)**.** Let

$$p(r) = 3 - 2r + 2r^2 - r^3.$$

Then $p(r) \geq 1$ for all $r \in [0,1]$.

Route (a). Regroup:

$$p(r) = 2(1-r) + 1 + r^2(2-r).$$

(Check: $2 - 2r + 1 + 2r^2 - r^3 = 3 - 2r + 2r^2 - r^3$.) On $[0,1]$ each of the three summands is nonnegative, the first because $r \leq 1$ and the third because $r \geq 0$ and $r \leq 1 < 2$, and the middle one equals 1. Hence $p \geq 1$ there, with equality only if $1 - r = 0$ and $r^2(2-r) = 0$, which is impossible; so $p > 1$ on $[0,1]$.

Route (b). Consider $p(r) - 2r^2 = 3 - 2r - r^3$. The function $r \mapsto 2r + r^3$ has derivative $2 + 3r^2 > 0$, so it is strictly increasing on $[0,1]$ and attains its maximum 3 at $r = 1$. Hence $3 - 2r - r^3 \geq 0$ on $[0,1]$, that is, $p(r) \geq 2r^2 \geq 0$.

The two bounds are incomparable: route (a) is positive at $r = 0$, where route (b) degenerates to $p \geq 0$, while route (b) is the better bound for $r > 1/\sqrt{2}$. Neither is sharp, since $p'(r) = -3r^2 + 4r - 2$ has discriminant $16 - 24 < 0$ and negative leading coefficient, so $p' < 0$, p is strictly decreasing, and $\min_{[0,1]} p = p(1) = 2$. Chapter 12 needs neither the sharp bound nor route (a): there p is multiplied by $r(1-r)$, which vanishes at $r = 0$, so $p \geq 2r^2$ suffices.

Corollary 3.12 (nonnegativity of K)**.** *For $r \in [0,1]$,*

$$K(r) = \frac{r(1-r)\,p(r)}{2(1+r^2)^2} \;\geq\; 0,$$

with $K(r) > 0$ on the open interval $(0,1)$ and $K(0) = K(1) = 0$.

Proof. The denominator $2(1+r^2)^2$ is positive for every real r. On $(0,1)$ we have $r > 0$, $1 - r > 0$, and $p(r) \geq 1 > 0$ by Example 3.11; the product of the three is positive. At $r = 0$ the factor r vanishes and at $r = 1$ the factor $1 - r$ vanishes. $\square$

Corollary 3.12 is the only inequality needed in the transitive dual certificate of Chapter 12; the rest of that verification is an identity between rational functions.

3.6 Symmetric combinations of three variables

Two elementary symmetric polynomials are needed, together with one identity relating them to the sum of squares and the substitution $u = 1 - U$.

Definition 3.13. For real x, y, z write

$$e_1 = x + y + z, \qquad e_2 = xy + yz + zx, \qquad e_3 = xyz.$$

Lemma 3.14. $x^2 + y^2 + z^2 = e_1^2 - 2e_2$.

Proof. $e_1^2 = (x+y+z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx) = x^2 + y^2 + z^2 + 2e_2$. $\square$

Lemma 3.15 (translating the constraint)**.** *Let $u, v, w > 0$ satisfy $uv + vw + wu = 1$. Put*

$$U = 1 - u, \qquad V = 1 - v, \qquad W = 1 - w, \qquad \tau = U + V + W.$$

Then

$$UV + VW + WU = 2(\tau - 1), \qquad \textit{and hence} \qquad U^2 + V^2 + W^2 = (\tau - 2)^2.$$

Proof. Substituting $u = 1 - U$ and so on into the constraint,

$$\begin{aligned} 1 &= (1-U)(1-V) + (1-V)(1-W) + (1-W)(1-U) \\ &= 3 - 2(U+V+W) + (UV+VW+WU), \end{aligned}$$

since each of U, V, W occurs in exactly two of the three products. Rearranging gives $UV + VW + WU = 2\tau - 2$. For the second identity apply Lemma 3.14 to (U, V, W), whose elementary symmetric functions are $e_1 = \tau$ and $e_2 = 2(\tau - 1)$:

$$U^2 + V^2 + W^2 = \tau^2 - 4(\tau - 1) = \tau^2 - 4\tau + 4 = (\tau - 2)^2. \qquad \square$$

Corollary 3.16. *If in addition $u, v, w \le 1$, so that $U, V, W \in [0, 1)$, then $1 \le \tau < 3$. In particular $\tau > 0$.*

Proof. For nonnegative U, V, W the quantity $UV + VW + WU$ is nonnegative, so $2(\tau - 1) \ge 0$ by Lemma 3.15, giving $\tau \ge 1$. Each of U, V, W is < 1 because the corresponding one of u, v, w is > 0, so $\tau < 3$. □

Strict positivity of τ also follows directly, and is what makes the denominator in Example 3.17 positive: if $U = V = W = 0$ then $u = v = w = 1$ and $uv + vw + wu = 3 \ne 1$, so the three variables cannot all equal 1.

Example 3.17 (the Chapter 11 denominator)**.** Under the hypotheses of Corollary 3.16,

$$V(u+v) + W(v+w) + U(w+u) \;=\; 4\tau - \tau^2 - 2.$$

Substituting $u = 1 - U$, $v = 1 - V$, $w = 1 - W$ turns $u + v$ into $2 - U - V$, and cyclically, so the left-hand side is

$$\begin{aligned} &V(2-U-V) + W(2-V-W) + U(2-W-U) \\ &\quad = 2\tau - (U^2+V^2+W^2) - (UV+VW+WU) \\ &\quad = 2\tau - (\tau-2)^2 - 2(\tau-1) \\ &\quad = 2\tau - \tau^2 + 4\tau - 4 - 2\tau + 2 \;=\; 4\tau - \tau^2 - 2, \end{aligned}$$

where the second step uses Lemma 3.15. In the first line the terms $-UV$, $-VW$, $-WU$ appear once each, and the terms $-V^2, -W^2, -U^2$ once each.

Two consequences.

(i) $\tau^2 - (4\tau - \tau^2 - 2) = 2\tau^2 - 4\tau + 2 = 2(\tau-1)^2 \ge 0$, so

$$4\tau - \tau^2 - 2 \;\le\; \tau^2,$$

with deficit $2(\tau - 1)^2$.

(ii) $4\tau - \tau^2 - 2 > 0$. Indeed it equals $V(u+v) + W(v+w) + U(w+u)$, a sum of three nonnegative terms; since $\tau > 0$ at least one of U, V, W is positive, and the corresponding factor $u + v$ (or $v + w$, or $w + u$) is positive, so at least one summand is positive.

Part (ii) can also be seen from Corollary 3.16: there $\tau \in [1, 3)$, and the quadratic $\tau \mapsto 4\tau - \tau^2 - 2$ has roots $2 \pm \sqrt{2}$ and opens downwards, so it is positive on $(2-\sqrt{2}, 2+\sqrt{2}) \supseteq [1, 3)$, since $2 - \sqrt{2} < 1$ and $3 < 2 + \sqrt{2}$.

Lemma 3.18 (at most one large variable)**.** *If $u, v, w > 0$ satisfy $uv + vw + wu = 1$, then at most one of u, v, w exceeds 1.*

Proof. Suppose two of them exceeded 1, say $u > 1$ and $v > 1$ (the other cases are identical). Then $uv > 1$. But all three summands uv, vw, wu are positive, so $uv < uv + vw + wu = 1$, a contradiction. □

Lemma 3.18 keeps the cyclic target from splitting into eight cases according to which of u, v, w exceed 1. Combined with the invariance of that target under the relabelling $(u, v, w) \mapsto (v, w, u)$, which permutes its three terms, it leaves two cases: all of u, v, w at most 1, or $u > 1 > v, w$.

3.7 Cauchy–Schwarz in Engel form

Theorem 3.19 (Engel form)**.** *Let $x_1, \dots, x_n$ be real and $y_1, \dots, y_n > 0$. Then*

$$\sum_{i=1}^{n} \frac{x_i^2}{y_i} \geq \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{\sum_{i=1}^{n} y_i},$$

with equality if and only if $x_1/y_1 = \cdots = x_n/y_n$.

Proof. First proof, from Cauchy–Schwarz. In $\mathbb{R}^n$ put

$$X = \Big(\frac{x_1}{\sqrt{y_1}}, \dots, \frac{x_n}{\sqrt{y_n}}\Big), \qquad Y = \big(\sqrt{y_1}, \dots, \sqrt{y_n}\big),$$

which is legitimate because every $y_i > 0$. Then $\langle X, Y\rangle = \sum_i x_i$, $\|X\|^2 = \sum_i x_i^2/y_i$ and $\|Y\|^2 = \sum_i y_i$, so Cauchy–Schwarz gives

$$\Big(\sum_i x_i\Big)^2 \leq \Big(\sum_i \frac{x_i^2}{y_i}\Big)\Big(\sum_i y_i\Big),$$

and dividing by the positive number $\sum_i y_i$ gives the inequality. Equality in Cauchy–Schwarz holds exactly when X and Y are linearly dependent; since $Y \neq 0$, this means $X = \mu Y$ for some real μ, that is, $x_i = \mu y_i$ for every i, that is, all the ratios x_i/y_i agree.

Second proof, by induction from the two-term case. For $n = 2$ we compute the difference over the common denominator $y_1y_2(y_1 + y_2) > 0$:

$$\frac{x_1^2}{y_1} + \frac{x_2^2}{y_2} - \frac{(x_1+x_2)^2}{y_1+y_2} = \frac{x_1^2 y_2(y_1+y_2) + x_2^2 y_1(y_1+y_2) - (x_1+x_2)^2 y_1 y_2}{y_1y_2(y_1+y_2)}.$$

The numerator expands to

$$x_1^2y_1y_2 + x_1^2y_2^2 + x_2^2y_1^2 + x_2^2y_1y_2 - (x_1^2 + 2x_1x_2 + x_2^2)y_1y_2 = (x_1y_2 - x_2y_1)^2,$$

so

$$\frac{x_1^2}{y_1} + \frac{x_2^2}{y_2} - \frac{(x_1+x_2)^2}{y_1+y_2} = \frac{(x_1y_2 - x_2y_1)^2}{y_1y_2(y_1+y_2)} \geq 0, \tag{3.1}$$

with equality exactly when $x_1y_2 = x_2y_1$, that is, $x_1/y_1 = x_2/y_2$.

Now induct on n, the case $n = 1$ being trivial. Assume the statement for $n - 1$ terms. Then

$$\sum_{i=1}^{n} \frac{x_i^2}{y_i} = \sum_{i=1}^{n-1} \frac{x_i^2}{y_i} + \frac{x_n^2}{y_n} \geq \frac{\left(\sum_{i<n} x_i\right)^2}{\sum_{i<n} y_i} + \frac{x_n^2}{y_n} \geq \frac{\left(\sum_{i\leq n} x_i\right)^2}{\sum_{i\leq n} y_i},$$

the last step being (3.1) applied to the pair $\big(\sum_{i<n} x_i,\, x_n\big)$ over $\big(\sum_{i<n} y_i,\, y_n\big)$, both denominators being positive.

Equality in the chain forces equality at both steps. By the inductive hypothesis the first gives $x_1/y_1 = \cdots = x_{n-1}/y_{n-1}$, with common value μ say; summing numerators and denominators, $\big(\sum_{i<n} x_i\big)/\big(\sum_{i<n} y_i\big) = \mu$ as well. The second then gives $x_n/y_n = \mu$. So all n ratios agree, and conversely equal ratios give equality. This recovers the equality case independently of the first proof. □

The result is also known as *Titu's lemma* and as *Sedrakyan's inequality.*

Remark 3.20 (the squaring trick). Theorem 3.19 bounds a sum of fractions whose numerators are squares. A sum of fractions x_i/c_i is not of that shape, but it can be made so: when $x_i > 0$,

$$\frac{x_i}{c_i} = \frac{x_i^2}{x_i c_i},$$

and Theorem 3.19 applies with numerators x_i and denominators $y_i = x_i c_i$, giving

$$\sum_i \frac{x_i}{c_i} \;\ge\; \frac{\left(\sum_i x_i\right)^2}{\sum_i x_i c_i}.$$

Both uses of the hypothesis $x_i > 0$ matter: it licenses the rewriting, and it makes $y_i = x_i c_i$ positive.

The denominator of the resulting bound is the weighted sum $\sum_i x_i c_i$ rather than $\sum_i c_i$, and a symmetric constraint can often evaluate that combination in closed form. In Example 3.17 it reduces to a quadratic in the single variable τ. The unweighted sum $\sum_i c_i = 2(u+v+w) = 2(3-\tau)$ is also a function of τ there, but the bound $\tau^2/\bigl(2(3-\tau)\bigr)$ built from it exceeds 1 only for $\tau \ge \sqrt{7}-1 \approx 1.6458$, and so is too weak on part of the range $\tau \in [1,3)$ (Exercise 3.3(c)), whereas $\tau^2/(4\tau-\tau^2-2) \ge 1$ holds throughout it (Proposition 3.22). In any case $\sum_i c_i$ is not a denominator that Theorem 3.19 delivers: the theorem produces $\sum_i y_i$, which after the rewriting is $\sum_i x_i c_i$ and nothing else.

Proposition 3.21 (Engel with vanishing numerators)**.** *Let $x_1, \dots, x_n \ge 0$, not all zero, and let $c_1, \dots, c_n > 0$. Then $\sum_i x_i c_i > 0$ and*

$$\sum_{i=1}^{n} \frac{x_i}{c_i} \;\ge\; \frac{\left(\sum_i x_i\right)^2}{\sum_i x_i c_i}.$$

Proof. Let $I = \{i : x_i > 0\}$, which is nonempty by hypothesis. Theorem 3.19 cannot be applied to the full index set, since after the squaring trick the denominator $y_i = x_i c_i$ vanishes whenever $x_i = 0$; hence the restriction to I. For $i \in I$ we have $x_i c_i > 0$, so the theorem applies over I with numerators x_i and denominators $y_i = x_i c_i$:

$$\sum_{i\in I} \frac{x_i}{c_i} = \sum_{i\in I} \frac{x_i^2}{x_i c_i} \;\ge\; \frac{\left(\sum_{i\in I} x_i\right)^2}{\sum_{i\in I} x_i c_i}.$$

Each of the three sums over I agrees with the corresponding sum over all indices: an omitted term has $x_i = 0$ and so contributes 0 to x_i/c_i, to $\sum_i x_i$ and to $\sum_i x_i c_i$. Positivity of $\sum_i x_i c_i = \sum_{i\in I} x_i c_i$ follows, I being nonempty. ☐

Some of the x_i can vanish in the application, which is the following estimate.

Proposition 3.22 (the τ-estimate)**.** *Let $u, v, w \in (0,1]$ satisfy $uv+vw+wu = 1$, and set $U = 1-u$, $V = 1-v$, $W = 1-w$, $\tau = U+V+W$. Then*

$$\frac{V}{u+v} + \frac{W}{v+w} + \frac{U}{w+u} \;\ge\; \frac{\tau^2}{4\tau-\tau^2-2} \;\ge\; 1.$$

Proof. By hypothesis $U, V, W \in [0,1)$, and $\tau > 0$ by Corollary 3.16; in particular U, V, W are not all zero. The three quantities $u+v$, $v+w$, $w+u$ are positive. Apply Proposition 3.21 with

$$(x_1, x_2, x_3) = (V, W, U), \qquad (c_1, c_2, c_3) = (u+v,\ v+w,\ w+u).$$

Its numerator is $(U+V+W)^2 = \tau^2$ and its denominator is

$$V(u+v) + W(v+w) + U(w+u) = 4\tau - \tau^2 - 2$$

by Example 3.17, which is positive (either by Proposition 3.21 itself, or by part (ii) of that example). This proves the first inequality. For the second, part (i) of Example 3.17 gives

$$\tau^2 - (4\tau - \tau^2 - 2) = 2(\tau-1)^2 \;\ge\; 0,$$

so $\tau^2 \ge 4\tau - \tau^2 - 2 > 0$, and dividing by the positive denominator (Lemma 3.5) gives $\tau^2/(4\tau - \tau^2 - 2) \ge 1$. ☐

Case 1 of Chapter 11 is this proposition together with the geometric identification of U, V, W. Exercise 3.3 supplies the identity $1 + u = (u+v) + V$ that connects the two.

3.8 Concave functions

This section and the next two concern functions of one real variable on an interval. Convex *sets*, half-planes, hulls and support functions belong to Chapter 4 and are not used here.

Definition 3.23. Let $I \subseteq \mathbb{R}$ be an interval. A function $\psi\colon I \to \mathbb{R}$ is *concave* if

$$\psi\big(\theta x + (1-\theta)y\big) \;\ge\; \theta\psi(x) + (1-\theta)\psi(y) \qquad \text{for all } x, y \in I,\ \theta \in [0,1],$$

and *strictly concave* if the inequality is strict whenever $x \neq y$ and $\theta \in (0,1)$. It is *convex* if $-\psi$ is concave.

Proposition 3.24 (derivative test)**.** *Let ψ be differentiable on an interval I with ψ' nonincreasing. Then ψ is concave. If ψ' is strictly decreasing then ψ is strictly concave. In particular, if ψ is twice differentiable with $\psi'' \le 0$ then ψ is concave, and $\psi'' < 0$ gives strict concavity.*

Proof. Let $x < y$ in I and $\theta \in (0,1)$ (the cases $\theta \in \{0,1\}$ and $x = y$ are trivial), and put $z = \theta x + (1-\theta)y$, so that $x < z < y$ and

$$z - x = (1-\theta)(y-x), \qquad y - z = \theta(y-x).$$

By the mean value theorem there are $\xi_1 \in (x,z)$ and $\xi_2 \in (z,y)$ with

$$\psi(z) - \psi(x) = \psi'(\xi_1)(z-x), \qquad \psi(y) - \psi(z) = \psi'(\xi_2)(y-z).$$

Since $\xi_1 < z < \xi_2$ and ψ' is nonincreasing, $\psi'(\xi_1) \ge \psi'(\xi_2)$. Hence

$$\begin{aligned}
\theta\psi(x) + (1-\theta)\psi(y) - \psi(z) &= (1-\theta)\big(\psi(y) - \psi(z)\big) - \theta\big(\psi(z) - \psi(x)\big)\\
&= (1-\theta)\psi'(\xi_2)\theta(y-x) - \theta\psi'(\xi_1)(1-\theta)(y-x)\\
&= \theta(1-\theta)(y-x)\big(\psi'(\xi_2) - \psi'(\xi_1)\big) \;\le\; 0,
\end{aligned}$$

which is the required inequality. If ψ' is strictly decreasing the last expression is strictly negative, since $\theta(1-\theta)(y-x) > 0$. The final claims follow by applying the mean value theorem to ψ', which converts the sign of ψ'' into monotonicity of ψ'. ☐

The hypothesis is stated as "ψ' nonincreasing" rather than "$\psi'' \le 0$" because §3.10 applies the proposition to a function glued from two pieces, which is differentiable everywhere but whose derivative is not differentiable at the join.

Proposition 3.25 (chord form and tangent form)**.** *Let ψ be concave on an interval I.*

(i) (Three-chord inequality.) For $x < y < z$ in I,

$$\frac{\psi(y) - \psi(x)}{y - x} \;\ge\; \frac{\psi(z) - \psi(x)}{z - x} \;\ge\; \frac{\psi(z) - \psi(y)}{z - y};$$

in particular the chord slope $(\psi(y) - \psi(x))/(y-x)$ is nonincreasing in each variable separately.

(ii) If moreover ψ is differentiable, then $\psi(y) \le \psi(x) + \psi'(x)(y-x)$ for all $x, y \in I$: the graph lies below every tangent line.

Proof. (i) Write $y = \theta x + (1-\theta)z$ with $\theta = (z-y)/(z-x) \in (0,1)$, so $1 - \theta = (y-x)/(z-x)$. Concavity gives $\psi(y) \ge \theta\psi(x) + (1-\theta)\psi(z)$. Subtracting $\psi(x)$,

$$\psi(y) - \psi(x) \;\ge\; (1-\theta)\big(\psi(z) - \psi(x)\big) = \frac{y-x}{z-x}\big(\psi(z) - \psi(x)\big),$$

and dividing by $y - x > 0$ gives the first inequality. Subtracting $\psi(z)$ instead,

$$\psi(y) - \psi(z) \;\ge\; \theta\big(\psi(x) - \psi(z)\big) = -\frac{z-y}{z-x}\big(\psi(z) - \psi(x)\big),$$

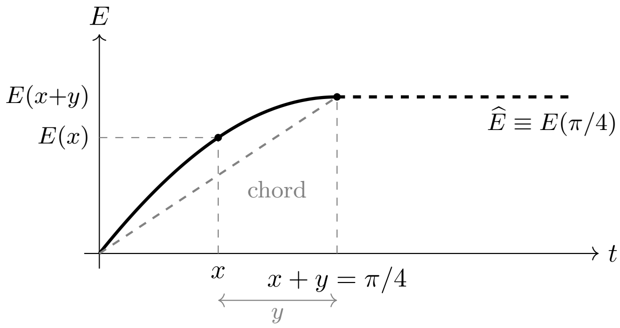


Figure 3.1: The threshold excess $E(t) = (\cos t + \sin t - 1)/2$ on $[0, \pi/4]$, with the vertical axis stretched. The graph lies above the grey dashed chord from the origin, which is the inequality $E(x) \geq \frac{x}{x+y} E(x+y)$ used in Theorem 3.27; the abscissae are x and $x+y$, with $x+y = \pi/4$ in the case drawn. The dashed horizontal continuation past $t = \pi/4$ is the extension $\widehat{E}$ of §3.10, which leaves the graph tangentially since $E'(\pi/4) = 0$.

that is, $\psi(z) - \psi(y) \leq \frac{z-y}{z-x}(\psi(z) - \psi(x))$; dividing by $z - y > 0$ gives the second.

(ii) Fix $x \neq y$ in I and let $\theta \in (0,1)$. Since $x + \theta(y-x) = (1-\theta)x + \theta y \in I$, concavity gives

$$\psi\big(x + \theta(y-x)\big) \;\geq\; (1-\theta)\psi(x) + \theta\psi(y),$$

that is,

$$\frac{\psi\big(x + \theta(y-x)\big) - \psi(x)}{\theta} \;\geq\; \psi(y) - \psi(x).$$

As $\theta \to 0^+$ the left-hand side tends to $\psi'(x)(y-x)$, by differentiability of ψ at x. Hence $\psi'(x)(y-x) \geq \psi(y) - \psi(x)$, which is the claim. For $x = y$ it is an equality. $\square$

Only Definition 3.23 itself is used below, through the convex-combination argument of §3.9. Parts (i) and (ii) are the standard working characterisations of concavity; part (i) is used in Exercise 3.6.

Example 3.26 (the threshold excess E)**.** Let

$$E(t) = H(t) - 1 = \frac{\cos t + \sin t - 1}{2}, \qquad t \in [0, \pi/4],$$

where $H(t) = (1 + \cos t + \sin t)/2$ is the owned-branch threshold function of Chapter 5; E is called the *threshold excess*. Then

$$E(0) = 0, \qquad E'(t) = \frac{\cos t - \sin t}{2}, \qquad E''(t) = -\frac{\cos t + \sin t}{2}.$$

On $[0, \pi/4]$ we have $\cos t \geq \sin t$ with equality only at $t = \pi/4$, so $E' \geq 0$ with $E'(\pi/4) = 0$; and $\cos t + \sin t \geq 1 > 0$ there, so $E'' < 0$. Hence, by Proposition 3.24, E is strictly concave on $[0, \pi/4]$; it is increasing, hence nonnegative, with

$$E(\pi/4) = \frac{\sqrt{2} - 1}{2} \approx 0.2071.$$

E' vanishes at $\pi/4$ because $\cos + \sin$ is invariant under $t \mapsto \pi/2 - t$, whose fixed point is $\pi/4$. Section 3.10 uses this. Chapter 5 records the same properties with their geometric meaning.

3.9 Subadditivity

Theorem 3.27 (subadditivity)**.** *Let $\psi\colon [0, \infty) \to \mathbb{R}$ be concave with $\psi(0) \geq 0$. Then*

$$\psi(x+y) \;\leq\; \psi(x) + \psi(y) \qquad \textit{for all } x, y \geq 0.$$

Proof. If $x = y = 0$ the assertion is $\psi(0) \le 2\psi(0)$, which holds because $\psi(0) \ge 0$. Otherwise $x + y > 0$, and we may write x as a convex combination of $x + y$ and 0:

$$x = \frac{x}{x+y}(x+y) + \frac{y}{x+y} \cdot 0.$$

Concavity and $\psi(0) \ge 0$ give

$$\psi(x) \;\ge\; \frac{x}{x+y}\psi(x+y) + \frac{y}{x+y}\psi(0) \;\ge\; \frac{x}{x+y}\psi(x+y),$$

and symmetrically $\psi(y) \ge \frac{y}{x+y}\psi(x+y)$. Adding the two inequalities gives $\psi(x) + \psi(y) \ge \psi(x + y)$. □

Remark 3.28 (which hypotheses do what). Subadditivity needs only concavity and $\psi(0) \ge 0$; monotonicity is used only for the comparison step of Corollary 3.29. If ψ is concave and nondecreasing with $\psi(0) = 0$ then $\psi \ge 0$ automatically, so the customary phrase "nonnegative increasing concave function vanishing at 0" is redundant in its first adjective.

Corollary 3.29 (subadditivity under a triangle inequality)**.** *Let $\psi\colon [0,\infty) \to \mathbb{R}$ be concave and nondecreasing with $\psi(0) = 0$. If $x, y, z \ge 0$ satisfy $x \le y + z$, then*

$$\psi(x) \;\le\; \psi(y) + \psi(z).$$

Proof. Since $x \le y + z$ and ψ is nondecreasing, $\psi(x) \le \psi(y + z)$; and $\psi(y + z) \le \psi(y) + \psi(z)$ by Theorem 3.27, whose hypotheses hold since $\psi(0) = 0 \ge 0$. □

Corollary 3.29 is the abstract form of the frame-excess inequality $A+1 \le B+C$ of Chapter 12. There ψ is the threshold excess E of Example 3.26, suitably extended as in §3.10, and x, y, z are three frame distances, which satisfy a triangle inequality because d_4 is a metric on the frame circle. The geometry belongs to Chapter 12; the analytic content is the corollary above.

Example 3.30 (three examples)**.** *(a)* $\psi(x) = \sqrt{x}$ on $[0,\infty)$ satisfies $\psi(0) = 0$ and is concave. Concavity may be checked straight from Definition 3.23, which avoids the failure of differentiability at 0: for $x, y \ge 0$ and $\theta \in (0, 1)$, both sides of $\sqrt{\theta x + (1-\theta)y} \ge \theta\sqrt{x} + (1-\theta)\sqrt{y}$ are nonnegative, so the inequality is equivalent to the one obtained by squaring, namely

$$\theta x + (1-\theta)y \;\ge\; \theta^2 x + (1-\theta)^2 y + 2\theta(1-\theta)\sqrt{xy},$$

that is, after subtracting and dividing by $\theta(1-\theta) > 0$, to $x + y \ge 2\sqrt{xy}$, which is Lemma 3.1. Theorem 3.27 therefore gives $\sqrt{x+y} \le \sqrt{x} + \sqrt{y}$.

(b) $\psi(x) = x-1$ is concave, being affine, but $\psi(0) = -1 < 0$; and $\psi(1+1) = 1 > 0 = \psi(1)+\psi(1)$. So the hypothesis $\psi(0) \ge 0$ cannot be dropped.

(c) $\psi(x) = x^2$ satisfies $\psi(0) = 0$ and is increasing on $[0,\infty)$, but is convex, not concave; and $\psi(1 + 1) = 4 > 2 = \psi(1) + \psi(1)$. So concavity cannot be dropped either.

3.10 Extending past a critical point

The function E of Example 3.26 is defined on $[0, \pi/4]$, but Corollary 3.29 needs a function on all of $[0,\infty)$, since a sum $y + z$ of two frame distances can be as large as $\pi/2$. The extension used is by a constant, and the following lemma treats the general case.

Lemma 3.31 (gluing)**.** *Let $a > 0$ and let $\psi\colon [0, a] \to \mathbb{R}$ be differentiable with ψ' nonincreasing and $\psi'(a) = 0$. Define $\widehat{\psi}\colon [0,\infty) \to \mathbb{R}$ by*

$$\widehat{\psi}(x) = \begin{cases} \psi(x), & 0 \le x \le a, \\ \psi(a), & x \ge a. \end{cases}$$

Then:

(1) $\widehat{\psi}$ *is differentiable on* $[0,\infty)$*, with* $\widehat{\psi}' = \psi'$ *on* $[0,a]$ *and* $\widehat{\psi}' = 0$ *on* $[a,\infty)$*;*

(2) $\widehat{\psi}'$ *is continuous wherever* ψ' *is, so* $\widehat{\psi}$ *is* C^1 *if* ψ *is;*

(3) $\widehat{\psi}'$ *is nonincreasing on* $[0,\infty)$*, so* $\widehat{\psi}$ *is concave;*

(4) $\widehat{\psi}' \geq 0$ *on* $[0,\infty)$*, so* $\widehat{\psi}$ *is nondecreasing;*

(5) $\widehat{\psi}(0) = \psi(0)$*.*

Proof. (1) On $[0,a)$ and on (a,∞) the function agrees with a differentiable function near each point, so only $x = a$ needs comment. There the left-hand derivative is $\psi'(a) = 0$ and the right-hand derivative is 0, since $\widehat{\psi}$ is constant to the right of a; the two agree, so $\widehat{\psi}$ is differentiable at a with $\widehat{\psi}'(a) = 0 = \psi'(a)$. The stated formula for $\widehat{\psi}'$ follows, and it is consistent at $x = a$.

(2) Immediate from the formula in (1), the two one-sided limits at a being $\lim_{x\to a^-} \psi'(x)$ and $0 = \psi'(a)$.

(3) The derivative $\widehat{\psi}'$ is nonincreasing on $[0,a]$ by hypothesis, is constantly 0 on $[a,\infty)$, and the two descriptions agree in value at a; so for $x \leq a \leq y$ we have $\widehat{\psi}'(x) \geq \widehat{\psi}'(a) = 0 = \widehat{\psi}'(y)$, and $\widehat{\psi}'$ is nonincreasing on $[0,\infty)$. Concavity follows from Proposition 3.24, whose hypothesis is on $\widehat{\psi}'$ rather than on a second derivative; $\widehat{\psi}'$ need not be differentiable at a.

(4) On $[0,a]$, ψ' is nonincreasing with $\psi'(a) = 0$, so $\psi' \geq 0$ there; on $[a,\infty)$, $\widehat{\psi}' = 0$. A differentiable function with nonnegative derivative on an interval is nondecreasing, by the mean value theorem.

(5) Immediate. □

Remark 3.32 (the role of $\psi'(a) = 0$). The hypothesis $\psi'(a) = 0$ makes $\widehat{\psi}$ differentiable at the join, so that Proposition 3.24 applies. With $\psi'(a) > 0$ the function $\widehat{\psi}$ has a corner at a, its one-sided derivatives there being $\psi'(a)$ and 0; the extension is still concave, since dropping the slope to 0 is a decrease, but establishing that requires a gluing argument for concavity across a corner, which is not needed here.

Example 3.33 (the extension used in Chapter 12)**.** Extend the threshold excess E from $[0,\pi/4]$ to $[0,\infty)$ by the constant value $E(\pi/4) = (\sqrt{2}-1)/2$:

$$\widehat{E}(x) = \begin{cases} \dfrac{\cos x + \sin x - 1}{2}, & 0 \leq x \leq \pi/4, \\[2ex] \dfrac{\sqrt{2}-1}{2}, & x \geq \pi/4. \end{cases}$$

By Example 3.26 the hypotheses of Lemma 3.31 hold with $a = \pi/4$: E is differentiable, E' is nonincreasing (indeed $E'' < 0$), and $E'(\pi/4) = 0$. So $\widehat{E}$ is C^1, nondecreasing, concave, and $\widehat{E}(0) = 0$. Corollary 3.29 therefore applies: whenever $x \leq y + z$ with $x, y, z \geq 0$,

$$\widehat{E}(x) \;\leq\; \widehat{E}(y) + \widehat{E}(z). \tag{3.2}$$

An extension is needed because frame distances lie in $[0,\pi/4]$ but their sums can reach $\pi/2$, outside the domain of E. The left-hand side of (3.2) is evaluated at a frame distance, so the values that matter are unchanged, but the intermediate step $\widehat{E}(x) \leq \widehat{E}(y+z)$ needs the larger domain.

Extending by the formula $(\cos x + \sin x - 1)/2$ instead would fail. That formula takes the value 0 at $x = \pi/2$, below its value $(\sqrt{2}-1)/2$ at $\pi/4$, so the formula-extension is not nondecreasing and Corollary 3.29 does not apply to it; and (3.2) fails for it, as $x = \pi/4$, $y = z = \pi/2$ shows, since then $x \leq y + z$ while the formula-extension gives $(\sqrt{2}-1)/2 > 0 + 0$.

3.11 Weierstrass substitutions as a coordinate system

In Part IA the substitution $r = \tan(t/2)$ appears as a device for integrating rational functions of sin and cos. Here it is used instead as a change of coordinates on a geometric parameter, converting trigonometric quantities into rational functions to which the methods of §§3.3–3.4 apply.

Proposition 3.34 (tangent form)**.** *Let $t \in (-\pi, \pi)$ and $r = \tan(t/2)$. Then*

$$\cos t = \frac{1 - r^2}{1 + r^2}, \qquad \sin t = \frac{2r}{1 + r^2},$$

and $t \mapsto r$ is a strictly increasing bijection $(-\pi, \pi) \to \mathbb{R}$, restricting to a strictly increasing bijection $(0, \pi/2) \to (0, 1)$.

Proof. Since $t/2 \in (-\pi/2, \pi/2)$ we have $\cos(t/2) \neq 0$, so r is defined, and

$$1 = \cos^2(t/2) + \sin^2(t/2) = \big(1 + \tan^2(t/2)\big)\cos^2(t/2) = (1 + r^2)\cos^2(t/2),$$

that is, $\cos^2(t/2) = 1/(1 + r^2)$. The double-angle formulas now give

$$\cos t = \cos^2(t/2) - \sin^2(t/2) = (1 - r^2)\cos^2(t/2) = \frac{1 - r^2}{1 + r^2},$$

$$\sin t = 2\sin(t/2)\cos(t/2) = 2r\cos^2(t/2) = \frac{2r}{1 + r^2}.$$

Finally $t \mapsto t/2$ is a strictly increasing bijection $(-\pi, \pi) \to (-\pi/2, \pi/2)$ and tan is a strictly increasing bijection $(-\pi/2, \pi/2) \to \mathbb{R}$; composing gives the stated bijection, and $t \in (0, \pi/2)$ corresponds to $t/2 \in (0, \pi/4)$, hence to $r \in (0, 1)$. □

Proposition 3.35 (cotangent form)**.** *Let $\gamma \in (0, \pi)$ and $q = \cot(\gamma/2)$. Then*

$$\sin\gamma = \frac{2q}{1 + q^2}, \qquad \cos\gamma = \frac{q^2 - 1}{1 + q^2},$$

and $\gamma \mapsto q$ is a strictly decreasing bijection $(0, \pi) \to (0, \infty)$.

Proof. Here $\gamma/2 \in (0, \pi/2)$, so $\sin(\gamma/2) \neq 0$ and $q > 0$ is defined. As above,

$$1 = \big(1 + \cot^2(\gamma/2)\big)\sin^2(\gamma/2) = (1 + q^2)\sin^2(\gamma/2),$$

so $\sin^2(\gamma/2) = 1/(1 + q^2)$, and

$$\sin\gamma = 2\sin(\gamma/2)\cos(\gamma/2) = 2q\sin^2(\gamma/2) = \frac{2q}{1 + q^2},$$

$$\cos\gamma = \cos^2(\gamma/2) - \sin^2(\gamma/2) = (q^2 - 1)\sin^2(\gamma/2) = \frac{q^2 - 1}{1 + q^2}.$$

Monotonicity: $\cot = 1/\tan$ is strictly decreasing on $(0, \pi/2)$ with range $(0, \infty)$, and $\gamma \mapsto \gamma/2$ is an increasing bijection $(0, \pi) \to (0, \pi/2)$. □

Note the sign flip relative to Proposition 3.34: the numerator of cos is $q^2 - 1$ in the cotangent form, not $1 - q^2$. The sign differs because q decreases as γ increases.

Lemma 3.36 (the two sign regimes)**.** *Let $\gamma \in (0, \pi)$ and $q = \cot(\gamma/2) > 0$. Then*

$$q \le 1 \iff \gamma \ge \pi/2 \iff \cos\gamma \le 0,$$

and consequently $|\cos\gamma| = |q^2 - 1|/(1 + q^2)$ and

$$\frac{1 + |\cos\gamma| + \sin\gamma}{2} = \Phi(q) := \frac{(1 + q)\max\{1, q\}}{1 + q^2}.$$

Proof. By Proposition 3.35 the map $\gamma \mapsto q$ is strictly decreasing, and $\cot(\pi/4) = 1$, so $q \le 1$ if and only if $\gamma/2 \ge \pi/4$, that is, $\gamma \ge \pi/2$. Also $\cos\gamma = (q^2 - 1)/(1 + q^2)$ has the sign of $q^2 - 1$, which is ≤ 0 exactly when $q \le 1$. This gives both equivalences and the formula for $|\cos\gamma|$.

For the displayed identity, two cases; in both, put everything over the common denominator $1 + q^2 > 0$ and use $\sin\gamma = 2q/(1 + q^2)$.

If $q \le 1$ then $|\cos\gamma| = (1-q^2)/(1+q^2)$, so

$$\frac{1+|\cos\gamma|+\sin\gamma}{2} = \frac{(1+q^2)+(1-q^2)+2q}{2(1+q^2)} = \frac{2+2q}{2(1+q^2)} = \frac{1+q}{1+q^2},$$

and $\max\{1,q\} = 1$, so this is $\Phi(q)$.

If $q \ge 1$ then $|\cos\gamma| = (q^2-1)/(1+q^2)$, so

$$\frac{1+|\cos\gamma|+\sin\gamma}{2} = \frac{(1+q^2)+(q^2-1)+2q}{2(1+q^2)} = \frac{2q^2+2q}{2(1+q^2)} = \frac{q(1+q)}{1+q^2},$$

and $\max\{1,q\} = q$, so again this is $\Phi(q)$. The two computations agree at $q = 1$, where both give 1. $\square$

The absolute value in Lemma 3.36 produces the max in Φ, and hence the two-case structure of the proof of the cyclic target. The case division reflects the geometry: an angle may be acute or obtuse.

Example 3.37 (properties of Φ)**.** *(a) Continuity at $q = 1$.* Both branches of Φ give $\Phi(1) = 2/2 = 1$, so the two formulas agree there and Φ is continuous on $(0,\infty)$, being a quotient of continuous functions on each of $(0,1]$ and $[1,\infty)$.

(b) The symmetry $\Phi(1/q) = \Phi(q)$. Suppose $q \ge 1$, so $1/q \le 1$ and $\max\{1,1/q\} = 1$. Then

$$\Phi(1/q) = \frac{(1+1/q)\cdot 1}{1+1/q^2} = \frac{q(q+1)}{q^2+1} = \Phi(q),$$

where we multiplied numerator and denominator by q^2 and used $\max\{1,q\} = q$. The case $q \le 1$ follows by exchanging the roles of q and $1/q$. The symmetry has a trigonometric source: by Proposition 3.35 and the identity $\cot((\pi-\gamma)/2) = \tan(\gamma/2)$, the substitution $q \mapsto 1/q$ corresponds to $\gamma \mapsto \pi - \gamma$, under which $\sin\gamma$ and $|\cos\gamma|$ are both unchanged, hence so is $(1+|\cos\gamma|+\sin\gamma)/2$.

(c) $\Phi(q) \ge 1$ for all $q > 0$, with equality if and only if $q = 1$. By cases, using Lemma 3.5 with the positive denominator $1+q^2$. For $q \le 1$: $\Phi(q) \ge 1$ if and only if $1+q \ge 1+q^2$, that is, $q \ge q^2$, that is, $q \le 1$ (dividing by $q > 0$), which holds, with equality at $q = 1$. For $q \ge 1$: $\Phi(q) \ge 1$ if and only if $q+q^2 \ge 1+q^2$, that is, $q \ge 1$, again with equality at $q = 1$.

Part (c) is the coordinate form of the threshold bound $H \ge 1$ of Chapter 5: by Lemma 3.36, $\Phi(q) = (1+|\cos\gamma|+\sin\gamma)/2$, so $\Phi \ge 1$ is the inequality $|\cos\gamma|+\sin\gamma \ge 1$, that is, $H \ge 1$.

Example 3.38 (the transitive dictionary)**.** Let $t \in (0,\pi/2)$, $r = \tan(t/2) \in (0,1)$, and write $c = \cos t$, $s = \sin t$. Then

$$\frac{1+c+s}{2} = \frac{1+r}{1+r^2}, \qquad \frac{1-c}{s} = r.$$

For the first, by Proposition 3.34,

$$\frac{1+c+s}{2} = \frac{1}{2}\cdot\frac{(1+r^2)+(1-r^2)+2r}{1+r^2} = \frac{2+2r}{2(1+r^2)} = \frac{1+r}{1+r^2}.$$

For the second,

$$\frac{1-c}{s} = \frac{\dfrac{(1+r^2)-(1-r^2)}{1+r^2}}{\dfrac{2r}{1+r^2}} = \frac{2r^2}{2r} = r,$$

using $r > 0$ to divide. (This is the identity $\tan(t/2) = (1-\cos t)/\sin t$, rederived in the coordinate.)

More generally, every quantity appearing in the transitive system becomes a rational function of r with denominator dividing $2(1+r)(1+r^2)^2$, which is positive on $(0,1)$. The verification of the certificate in Chapter 12 is therefore a polynomial identity rather than a trigonometric one.

Example 3.39 (deriving the certificate coefficients)**.** Solve, for α and β, the system

$$\alpha + (1+c)\beta = \frac{1+c+s}{2}, \qquad s\alpha + s(1-c)\beta = 1-c, \tag{3.3}$$

where $c = \cos t$, $s = \sin t$ and $t \in (0, \pi/2)$.

Since $s > 0$, the second equation may be divided by s, and by Example 3.38 it becomes

$$\alpha + (1-c)\beta = \frac{1-c}{s} = r.$$

Subtracting this from the first equation eliminates α:

$$\big((1+c)-(1-c)\big)\beta = 2c\beta = \frac{1+r}{1+r^2} - r = \frac{(1+r)-r(1+r^2)}{1+r^2} = \frac{1-r^3}{1+r^2}.$$

Now $2c = 2(1-r^2)/(1+r^2) = 2(1-r)(1+r)/(1+r^2)$, which is nonzero for $r \in (0,1)$, so

$$\beta = \frac{1-r^3}{1+r^2} \cdot \frac{1+r^2}{2(1-r)(1+r)} = \frac{1-r^3}{2(1-r)(1+r)} = \frac{1+r+r^2}{2(1+r)},$$

cancelling the factor $1-r > 0$ from $1-r^3 = (1-r)(1+r+r^2)$. Then, with $1-c = 2r^2/(1+r^2)$,

$$\alpha = r - (1-c)\beta = r - \frac{2r^2}{1+r^2} \cdot \frac{1+r+r^2}{2(1+r)} = r \cdot \frac{(1+r^2)(1+r) - r(1+r+r^2)}{(1+r^2)(1+r)}.$$

The bracket is $(1+r+r^2+r^3) - (r+r^2+r^3) = 1$, so

$$\alpha = \frac{r}{(1+r)(1+r^2)}.$$

These are the quantities named in the transitive target, and both are visibly positive for $r \in (0,1)$. Uniqueness of the solution follows from the elimination: the system is triangular after the two reductions, and $2c \neq 0$.

In the variables c and s the same system is unpleasant to solve; in r it reduces to the elimination above. Chapters 8 and 12 explain where (3.3) comes from and why its solution is called a dual certificate.

3.12 The cotangent addition formula and the supplement identity

This section converts a statement about a sum of angles into a polynomial constraint. Chapter 11 produces the angle statement by a winding argument.

Lemma 3.40 (addition)**.** *Whenever* $\sin x$*,* $\sin y$ *and* $\sin(x+y)$ *are all nonzero,*

$$\cot(x+y) = \frac{\cot x \cot y - 1}{\cot x + \cot y},$$

and in particular the denominator $\cot x + \cot y$ *is then nonzero.*

Proof. By the addition formulas,

$$\cot(x+y) = \frac{\cos(x+y)}{\sin(x+y)} = \frac{\cos x \cos y - \sin x \sin y}{\sin x \cos y + \cos x \sin y}.$$

Divide numerator and denominator by $\sin x \sin y \neq 0$: the numerator becomes $\cot x \cot y - 1$ and the denominator becomes $\cot y + \cot x$. The denominator of the original expression, $\sin(x+y)$, is nonzero by hypothesis, so after dividing by $\sin x \sin y$ the new denominator $\cot x + \cot y = \sin(x+y)/(\sin x \sin y)$ is nonzero as well. $\square$

Lemma 3.41 (supplement and complement)**.** *For every z at which the expressions are defined,*

$$\cot(\pi - z) = -\cot z, \qquad \cot\big(\tfrac{\pi}{2} - z\big) = \tan z.$$

Proof. $\sin(\pi - z) = \sin z$ and $\cos(\pi - z) = -\cos z$, so $\cot(\pi - z) = -\cos z/\sin z = -\cot z$. Likewise $\sin(\pi/2 - z) = \cos z$ and $\cos(\pi/2 - z) = \sin z$, so $\cot(\pi/2 - z) = \sin z/\cos z = \tan z$. ☐

Theorem 3.42 (angle sum as an algebraic constraint)**.** *Let $x, y, z \in (0, \pi/2)$ and put $u = \cot x$, $v = \cot y$, $w = \cot z$, all positive. Then*

$$x + y + z = \pi \iff uv + vw + wu = 1.$$

Proof. Throughout, $x + y \in (0, \pi)$ and $\pi - z \in (\pi/2, \pi)$, so $\sin(x + y) > 0$ and $\sin(\pi - z) > 0$; also $\sin x, \sin y > 0$. Thus Lemma 3.40 applies to x and y, and $u + v > 0$ in any case.

($\Rightarrow$) If $x + y + z = \pi$ then $x + y = \pi - z$, so by Lemma 3.41,

$$\frac{uv - 1}{u + v} = \cot(x + y) = \cot(\pi - z) = -w.$$

Multiplying by $u+v > 0$ (Lemma 3.5) gives $uv-1 = -w(u+v) = -wu-wv$, that is, $uv+vw+wu = 1$.

($\Leftarrow$) Conversely, suppose $uv+vw+wu = 1$. Then $uv-1 = -w(u+v)$, and dividing by $u+v > 0$ gives

$$\cot(x + y) = \frac{uv - 1}{u + v} = -w = -\cot z = \cot(\pi - z),$$

again by Lemma 3.41. Both $x + y$ and $\pi - z$ lie in $(0, \pi)$, on which cot is strictly decreasing and hence injective; therefore $x + y = \pi - z$, that is, $x + y + z = \pi$. ☐

Theorem 3.42 converts the angle-sum condition produced by the winding argument of Chapter 11 into the algebraic constraint $uv + vw + wu = 1$.

Corollary 3.43 (denominator trading)**.** *If $u, v, w > 0$ satisfy $uv + vw + wu = 1$, then*

$$(u + v)(u + w) = 1 + u^2,$$

together with the cyclic variants $(v + w)(v + u) = 1 + v^2$ and $(w + u)(w + v) = 1 + w^2$.

Proof. $(u + v)(u + w) = u^2 + uw + uv + vw = u^2 + (uv + vw + wu) = u^2 + 1$. The variants follow by relabelling, the constraint being symmetric. ☐

The half-angle formulas of §3.11 produce denominators of the form $1+q^2$, whereas the bound of §3.7 has denominator $\sum_i y_i$, a sum. Corollary 3.43 replaces each $1+u^2$ by the product $(u+v)(u+w)$ of two pairwise sums, and after cancellation the three terms of the cyclic target have denominators $u + v$, $v + w$, $w + u$, to which Theorem 3.19 applies. Chapter 11 opens its scalar section with this substitution.

Example 3.44 (the cyclic dictionary)**.** Let $\gamma_1, \gamma_2, \gamma_3 \in (0, \pi)$ satisfy $\gamma_1 + \gamma_2 + \gamma_3 = 2\pi$, and set

$$u = \cot(\gamma_1/2), \qquad v = \cot(\gamma_2/2), \qquad w = \cot(\gamma_3/2),$$

all positive by Proposition 3.35. The half-gaps $\gamma_i/2$ lie in $(0, \pi/2)$ and sum to π, so Theorem 3.42 gives

$$uv + vw + wu = 1.$$

Combining Proposition 3.35 with Corollary 3.43,

$$\sin\gamma_1 = \frac{2u}{1 + u^2} = \frac{2u}{(u + v)(u + w)},$$

and cyclically $\sin\gamma_2 = 2v/\big((v + w)(v + u)\big)$ and $\sin\gamma_3 = 2w/\big((w + u)(w + v)\big)$. Finally, by Lemma 3.36,

$$\frac{1 + |\cos\gamma_1| + \sin\gamma_1}{2} = \Phi(u),$$

and cyclically.

Chapter 11 supplies the geometry behind the hypothesis $\gamma_1+\gamma_2+\gamma_3 = 2\pi$, namely three directed gaps between unit normals winding once around the circle, and identifies $(1 + |\cos\gamma_i| + \sin\gamma_i)/2$ as a branch threshold.

Exercises

Exercise 3.1. (a) Prove that $u^2 + 1/u^2 \geq u + 1/u$ for every $u > 0$, and determine the equality case. [Set $y = u + 1/u$, express $u^2 + 1/u^2$ in terms of y, and use Corollary 3.2.] (b) Let $u, v > 0$ satisfy $u + v = 1$. Prove $\left(1 + \frac{1}{u}\right)\left(1 + \frac{1}{v}\right) \geq 9$ by the method of Remark 3.3, and determine when equality holds. [Put the product over the denominator uv and use the constraint to simplify the numerator before estimating anything.]

Exercise 3.2. (a) A student writes: "Since $(1+v)/(v+w) \geq (1+v)^2/(1+v^2)$, cross-multiplying gives $1 + v^2 \geq (v + w)(1 + v)$." Name the positivity assumptions under which Lemma 3.5 licenses the step. Then show that the implication is nonetheless true for all real v, w with $v + w \neq 0$, so that here the positivity hypotheses are dispensable. [Write $A = 1+v$, $B = v+w$, $D = 1+v^2 > 0$. Suppose the conclusion failed, so $AB > D > 0$; deduce that A and B have the same sign, and treat the two sign cases separately. Check also what happens in the mixed-sign cases, including $A = 0$.] Explain what the factor $1 + v$ shared by the two numerators contributes. (b) Contrast this with the pair: "from $1/(v+w) \geq (1+v)/(1+v^2)$ conclude $1+v^2 \geq (v+w)(1+v)$". Verify that $v = -2$, $w = -4$ satisfies the premise but not the conclusion, and identify which step of the argument in (a) breaks when the shared factor is removed. (c) Let $u, v, w > 0$ satisfy $uv + vw + wu = 1$ and suppose $u \leq 1$. Show that the inequality of Example 3.6 reverses: $(v + w)(1 + v) \geq 1 + v^2$, and hence $(1 + v)/(v + w) \leq (1 + v)^2/(1 + v^2)$, with equality exactly when $u = 1$. Deduce why the argument of Example 3.6 cannot be reused when all three variables are at most 1.

Exercise 3.3. Let $u, v, w \in (0, 1]$ satisfy $uv + vw + wu = 1$, and set $U = 1 - u$, $V = 1 - v$, $W = 1 - w$, $\tau = U + V + W$. (a) Verify the identity $1 + u = (u + v) + V$ and its two cyclic analogues, taking care with the index shift, and deduce from Proposition 3.22 that

$$\frac{1+u}{u+v} + \frac{1+v}{v+w} + \frac{1+w}{w+u} \;\geq\; 4.$$

(This is Case 1 of the cyclic target.) (b) Deduce, *without* using Theorem 3.19 or Proposition 3.21, that at least one of the three fractions $V/(u+v)$, $W/(v+w)$, $U/(w+u)$ is at least 1/3. [Suppose all three are $< 1/3$. Summing the three resulting inequalities and using $u + v + w = 3 - \tau$ gives an upper bound on τ; multiplying the i-th inequality by its numerator and summing, then using Lemma 3.15 and Example 3.17, gives a lower bound. Check that the two are incompatible.] (c) Remark 3.20 states that the unweighted denominator is too weak. Make this precise: show $\sum_i c_i = (u+v)+(v+w)+(w+u) = 2(3-\tau)$, show that $\tau^2/\big(2(3-\tau)\big) \geq 1$ if and only if $\tau \geq \sqrt{7}-1$, and verify that the symmetric solution $u = v = w = 1/\sqrt{3}$ has $\tau = 3 - \sqrt{3} < \sqrt{7} - 1$. Conclude that the bound of Proposition 3.22 cannot be obtained from the unweighted denominator.

Exercise 3.4. (a) Show that $1-2r+3r^2-r^3 \geq 0$ on $[0, 1]$ by regrouping it into visibly nonnegative pieces. Then show, by examining the derivative at the two endpoints, that its minimum on $[0, 1]$ is attained at an interior point, so that no regrouping can exhibit that minimum as a constant summand. (The value of the minimum is not wanted.) (b) Let $K(r) = r(1 - r)p(r)/\big(2(1 + r^2)^2\big)$ with $p(r) = 3 - 2r + 2r^2 - r^3$. Show that $K(r) \leq \frac{3}{2}r(1 - r)$ on $[0, 1]$, and deduce from Lemma 3.1 that $K \leq 3/8$ there. Explain why only the sign of K, and not its size, matters in Chapter 12.

Exercise 3.5. (a) Deduce from Theorem 3.19 the arithmetic–harmonic mean inequality: for $c_1, \ldots, c_n > 0$,

$$\sum_{i=1}^{n} \frac{1}{c_i} \;\geq\; \frac{n^2}{\sum_{i=1}^{n} c_i},$$

and identify the equality case. (b) For $a, b, c > 0$ prove $\dfrac{a^2}{b+c} + \dfrac{b^2}{c+a} + \dfrac{c^2}{a+b} \;\geq\; \dfrac{a+b+c}{2}$. (c) Use the squaring trick of Remark 3.20 to prove Nesbitt's inequality: for $a, b, c > 0$,

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \;\geq\; \frac{3}{2}.$$

[After applying the trick one needs $(a+b+c)^2 \geq 3(ab+bc+ca)$; prove it from Lemma 3.14.] Compare the route through (b), which reaches the same bound by a different application of Theorem 3.19, and say which of the two is the analogue of the argument used in Proposition 3.22. (d) Show that the hypothesis "not all zero" cannot be dropped from Proposition 3.21, and say which quantity in the statement is at fault.

Exercise 3.6. (a) Show that $\psi(x) = \log(1+x)$ is nondecreasing and concave on $[0,\infty)$ with $\psi(0) = 0$, and deduce that $\log(1+x+y) \le \log(1+x) + \log(1+y)$ for all $x, y \ge 0$. (b) Let $\psi(x) = x - x^2/2$ on $[0,1]$, extended by the constant $1/2$ on $[1,\infty)$. Verify the hypotheses of Lemma 3.31 and conclude that the extension $\widehat{\psi}$ is C^1, nondecreasing and concave; deduce that $\widehat{\psi}(x) \le \widehat{\psi}(y) + \widehat{\psi}(z)$ whenever $x \le y+z$ with $x, y, z \ge 0$. (c) Show by explicit computation that if instead ψ is extended to $[0,\infty)$ by the formula $x - x^2/2$, the conclusion of (b) fails: find $x \le y+z$ with $\psi(x) > \psi(y) + \psi(z)$, and identify which hypothesis of Corollary 3.29 has been lost. (d) Using Proposition 3.25(i), show that a concave ψ on $[0,\infty)$ with $\psi(0) \ge 0$ satisfies $\psi(x)/x \ge \psi(y)/y$ for $0 < x < y$, and give a second proof of Theorem 3.27 from this. (e) Let ψ and a be as in Lemma 3.31, and let $g\colon [0,\infty) \to \mathbb{R}$ be any concave function agreeing with ψ on $[0,a]$. Using Proposition 3.25(i) and the hypothesis $\psi'(a) = 0$, show that $g(x) \le \psi(a)$ for every $x \ge a$; so $\widehat{\psi}$ is the pointwise-largest concave extension of ψ. Deduce that it is the only concave nondecreasing one.

Exercise 3.7. (a) Compute the one-sided derivatives of Φ at $q = 1$ from its two branches, and check that they are $-\frac{1}{2}$ and $+\frac{1}{2}$; conclude that Φ has a corner, not a horizontal tangent, at its minimum. Show that this is consistent with the symmetry $\Phi(1/q) = \Phi(q)$ of Example 3.37(b), by differentiating the symmetry to see what it would force, and contrast with E at the fixed point $t = \pi/4$ of $t \mapsto \pi/2 - t$ in Example 3.26, where the corresponding involution does produce a horizontal tangent. (b) Show that $\Phi(q) \le (1+\sqrt{2})/2$ for all $q > 0$, with equality exactly at $q = 1+\sqrt{2}$ and $q = \sqrt{2}-1$, and interpret both the bound and the two equality points through Lemma 3.36. (c) With $t \in (0,\pi/2)$, $r = \tan(t/2)$, $c = \cos t$ and $s = \sin t$, add two entries to the dictionary of Example 3.38:

$$\frac{1+c-s}{2} = \frac{1-r}{1+r^2}, \qquad \frac{s}{1+c} = r,$$

and deduce the identity $\dfrac{1+c+s}{2} \cdot \dfrac{1+c-s}{2} = \dfrac{c}{1+r^2}$. (d) Let $\gamma_1, \gamma_2, \gamma_3 \in (0,\pi)$ sum to 2π, and let u, v, w be the half-gap cotangents of Example 3.44. Show that the only solution with $u = v = w$ is $u = v = w = 1/\sqrt{3}$, corresponding to $\gamma_1 = \gamma_2 = \gamma_3 = 2\pi/3$, and evaluate the left-hand side of the cyclic target there exactly. By how much does it exceed 4?

Exercise 3.8 (capstone)**.** Chapter 11 proves the following, which is Case 2 of the cyclic target. Let $u > 1$ and $0 < v, w < 1$ satisfy $uv + vw + wu = 1$. Prove

$$\frac{u(1+u)}{u+v} + \frac{1+v}{v+w} + \frac{1+w}{w+u} \;\ge\; 4.$$

[Two hints. First, only the middle fraction needs the constraint; use it in the form $u(v+w)+vw = 1$ to eliminate the constant 1 from a difference of denominators, and check every sign before dividing. Second, aim to reduce the problem to $u + 1/u \ge 2$ together with a single inequality in v alone.]

Then state which hypotheses are used where, and locate the place where $u > 1$, rather than $u > 0$, is needed. Finally, explain why the three terms of the cyclic target reduce to the three displayed fractions in this case: use Corollary 3.43 to trade denominators, and Lemma 3.18 to justify that the case division is exhaustive.

Chapter 4

Convex Sets, Support Functions, and Squares

This chapter assumes.

- From Chapter 1: the open ball $B(x,\rho)$ and the interior $\operatorname{int} X$; the frame $F=(e,f)$ and the unit square $Q(F,P)$; the description of $\operatorname{int} Q(F,P)$ by the two strict slab inequalities; the notion of disjoint interiors; the centre-separation lemma.
- From Chapter 2: the determinant $\det(u,v) = u_1v_2 - u_2v_1$; the quarter turn J_σ and the four facts $J_\sigma^2 = -I$, $\langle J_\sigma u, v\rangle = \sigma \det(u,v)$, $J_\sigma^\mathsf{T} = -J_\sigma$ and $\langle J_\sigma u, J_\sigma v\rangle = \langle u,v\rangle$ (Proposition 2.8(i),(ii),(iii),(v)); and the unit-vector companion $|a|+|b| \geq 1$ for the coordinates of a unit vector in an orthonormal basis.
- Nothing here uses compactness, closedness or continuity. Every maximum that is asserted below to be attained is either a maximum over a finite set or is attained at a point exhibited explicitly; the only suprema not asserted to be attained are those over open balls.

Every statement below holds for either choice of the sign $\sigma \in \{-1,+1\}$; the sign is not yet frozen. We write $\bar{B}(x,\rho) = \{y : \|y-x\| \leq \rho\}$ for the closed ball, $B(x,\rho)$ remaining the open one.

Three results are proved. First, that a point interior to a unit square satisfies a strict inequality in every direction (Proposition 4.15). Second, that the two squares $Q(F,P)$ and $Q(G,Q)$ have interiors meeting if and only if $Q-P$ lies in the interior of a fixed set Z (Theorem 4.29). Third, that this fixed set is cut out by at most eight half-planes whose normals are axes of the two frames: eight unless the two frames share their axes, when four suffice (Theorem 4.39). Chapters 10–12 use the third, in the form of the *ownership* of a separating normal by one of the two squares.

Letters. Generic convex sets in this chapter are X and Y; the two squares are $Q_F := Q(F,0)$ and $Q_G := Q(G,0)$; the obstacle is Z, and a generic zonogon is $\operatorname{Zon}(\dots)$; the auxiliary set inside the proof of Theorem 4.39 is W. Vertices of a triangle are V_1, V_2, V_3. Here λ, μ and θ denote weights in a convex combination; Chapter 8 lets λ_i denote, more generally, a coefficient in a conic combination.[1]

4.1 Convex sets, convex combinations, half-planes

Definition 4.1 (segment, convex set)**.** For $x,y\in\mathbb{R}^2$ the *segment* from x to y is

$$[x,y] = \{(1-\lambda)x + \lambda y \ : \ \lambda\in[0,1]\}.$$

A set $X\subseteq\mathbb{R}^2$ is *convex* if $[x,y]\subseteq X$ whenever $x,y\in X$.

The empty set and every singleton are convex.

[1] The letter h carrying a set as subscript, as in h_X, always denotes a support function (Section 4.7); a bare h remains an altitude, as in Chapter 2. The number $H = (1+c+s)/2$ of Section 4.8 is a branch threshold, written $H(t)$ in Chapter 5.

Definition 4.2 (convex combination)**.** A *convex combination* of $x_1, \ldots, x_m \in \mathbb{R}^2$ is a point $\sum_{i=1}^m \lambda_i x_i$ with $\lambda_i \geq 0$ for all i and $\sum_{i=1}^m \lambda_i = 1$.

Lemma 4.3. *A set $X \subseteq \mathbb{R}^2$ is convex if and only if it contains every convex combination of finitely many of its points.*

Proof. ($\Leftarrow$) A point of $[x, y]$ is a convex combination of the two points x, y.

($\Rightarrow$) Let X be convex; we prove by induction on $m \geq 1$ that every convex combination of m points of X lies in X. For $m = 1$ the combination is x_1. Suppose the claim holds for $m - 1$, and let $x = \sum_{i=1}^m \lambda_i x_i$ with $\lambda_i \geq 0$ and $\sum_i \lambda_i = 1$. If $\lambda_m = 1$ then all other λ_i vanish and $x = x_m \in X$. Otherwise $1 - \lambda_m > 0$, and

$$x = (1 - \lambda_m)\, y + \lambda_m x_m, \qquad y := \sum_{i=1}^{m-1} \frac{\lambda_i}{1 - \lambda_m}\, x_i.$$

The coefficients of y are non-negative and sum to $(1-\lambda_m)/(1-\lambda_m) = 1$, so $y \in X$ by the inductive hypothesis, and then $x \in [y, x_m] \subseteq X$ by convexity. $\square$

Definition 4.4 (half-plane, slab)**.** Let $n \in \mathbb{R}^2$ be nonzero and $c \in \mathbb{R}$. The *closed half-plane* with normal n and level c is $\{x : \langle x, n\rangle \leq c\}$. For $c \geq 0$, the *slab* $\{x : |\langle x, n\rangle| \leq c\}$ is the intersection of the two half-planes with normals n and $-n$, both at level c.

Lemma 4.5. *(i) Every half-plane, every slab, and every open or closed ball $B(x, \rho)$, $\bar{B}(x, \rho)$, is convex; so is $\mathbb{R}^2$.*
(ii) The intersection of any family of convex sets is convex.
(iii) If X is convex and $T(x) = Mx + b$ is affine (M a 2×2 matrix, $b \in \mathbb{R}^2$) then $T(X) = \{Mx + b : x \in X\}$ is convex.

Proof. (i) If $\langle x, n\rangle \leq c$ and $\langle y, n\rangle \leq c$ then, for $\lambda \in [0, 1]$, linearity gives $\langle (1 - \lambda)x + \lambda y, n\rangle = (1 - \lambda)\langle x, n\rangle + \lambda\langle y, n\rangle \leq (1 - \lambda)c + \lambda c = c$. A slab is an intersection of two half-planes, so (ii) applies. For a ball, if $\|x - z\| < \rho$ and $\|y - z\| < \rho$ then

$$\|(1 - \lambda)x + \lambda y - z\| = \|(1 - \lambda)(x - z) + \lambda(y - z)\| \leq (1 - \lambda)\|x - z\| + \lambda\|y - z\| < \rho,$$

and the same display with both strict inequalities replaced by $\leq$ handles $\bar{B}(z, \rho)$.

(ii) If x, y lie in every member of the family, so does every point of $[x, y]$.

(iii) $T\big((1 - \lambda)x + \lambda y\big) = (1 - \lambda)T(x) + \lambda T(y)$, because the coefficients sum to 1; so the image of a segment is a segment. $\square$

Definition 4.6 (convex polygon, half-plane form)**.** A *convex polygon* is a bounded set of the form

$$\Pi = \bigcap_{i=1}^{N} \{x : \langle x, n_i\rangle \leq c_i\}$$

with $N \geq 1$, each n_i a unit vector and each $c_i \in \mathbb{R}$. The n_i are *outward normals* of the presentation. For each i the set $\{x \in \Pi : \langle x, n_i\rangle = c_i\}$, when nonempty, is a *facet* of the presentation, possibly a single point; a facet containing more than one point is, in the plane, an *edge*. A *vertex* is a point of Π lying on two facets whose normals are linearly independent.

By Lemma 4.5 a convex polygon is convex. The definition describes a set by inequalities, which is the form in which every polygon below arises.

Being a facet is a property of the presentation, not of the set: adding the redundant inequality $x_1 + x_2 \leq 1$ to the presentation of $[0, \frac{1}{2}]^2$ produces a facet consisting of the single point $(\frac{1}{2}, \frac{1}{2})$. A presentation may also contain an inequality whose facet is empty. Neither degeneracy occurs below: in every presentation used here each facet is a segment.

Remark 4.7 (Minkowski–Weyl). A bounded intersection of finitely many half-planes is also the convex hull of finitely many points, and conversely the convex hull of finitely many points is a bounded intersection of finitely many half-planes. This is the *Minkowski–Weyl theorem* (in the plane, and in every dimension). It is stated for orientation and not used: on the two occasions where both descriptions of a particular polygon are needed, for a triangle in Example 4.11 and for the obstacle Z in Theorem 4.39, each description is verified directly.

Figure 4.1: Convexity is the statement that every chord stays inside. The grey dashed chords of the third and fourth sets leave them.

4.2 Convex hulls, and triangles by determinant signs

Definition 4.8. For $X \subseteq \mathbb{R}^2$, the *convex hull* $\operatorname{conv} X$ is the set of all convex combinations of finitely many points of X.

Lemma 4.9. $\operatorname{conv} X$ *is convex, contains* X*, and is contained in every convex set containing* X*. Hence* $\operatorname{conv} X$ *is the smallest convex set containing* X*, and* X *is convex if and only if* $X = \operatorname{conv} X$*.*

Proof. $X \subseteq \operatorname{conv} X$ because x is the convex combination with the single coefficient 1.

Convexity: let $p = \sum_i \lambda_i x_i$ and $q = \sum_j \mu_j y_j$ be convex combinations of points of X and let $\theta \in [0,1]$. Then $(1-\theta)p + \theta q = \sum_i (1-\theta)\lambda_i x_i + \sum_j \theta \mu_j y_j$ is a combination of points of X with non-negative coefficients summing to $(1-\theta)+\theta = 1$, hence lies in $\operatorname{conv} X$.

Minimality: if Y is convex and $X \subseteq Y$, then every convex combination of finitely many points of X is a convex combination of points of Y, hence lies in Y by Lemma 4.3; so $\operatorname{conv} X \subseteq Y$.

The final sentence: if $X = \operatorname{conv} X$ then X is convex; conversely if X is convex then $\operatorname{conv} X \subseteq X$ by minimality, and the reverse inclusion always holds. □

Lemma 4.10 (maxima over a hull of finitely many points)**.** *Let* $x_1, \ldots, x_m \in \mathbb{R}^2$ *and* $n \in \mathbb{R}^2$*. Then*

$$\max_{x \in \operatorname{conv}\{x_1,\ldots,x_m\}} \langle x, n\rangle = \max_{1 \le i \le m} \langle x_i, n\rangle,$$

and the maximum on the left is attained at any x_i *attaining the maximum on the right.*

Proof. For a convex combination $x = \sum_i \lambda_i x_i$, linearity gives

$$\langle x, n\rangle = \sum_i \lambda_i \langle x_i, n\rangle \le \Big(\sum_i \lambda_i\Big) \max_j \langle x_j, n\rangle = \max_j \langle x_j, n\rangle.$$

The value $\max_j \langle x_j, n\rangle$ is itself attained, at the corresponding x_j, which lies in the hull. □

This lemma computes the two support functions used below: that of a unit square (Proposition 4.31) and that of a segment, hence of a zonogon (Lemma 4.37). In both cases a maximum over an infinite set is replaced by a maximum over finitely many vertices, so no existence theorem is required.

Example 4.11 (a triangle in both forms)**.** Let $V_1, V_2, V_3 \in \mathbb{R}^2$ be non-collinear, put $u = V_2 - V_1$, $v = V_3 - V_1$ and

$$\Delta = \det(u, v) \ne 0.$$

Then

$$\operatorname{conv}\{V_1, V_2, V_3\} = \left\{ x \ :\ \begin{array}{ll} \dfrac{\det(u,\, x - V_1)}{\Delta} \ge 0, & \dfrac{\det(x - V_1,\, v)}{\Delta} \ge 0, \\ \dfrac{\det(V_3 - V_2,\, x - V_2)}{\Delta} \ge 0 & \end{array} \right\}.$$

Proof. Since $\Delta \ne 0$, the pair (u, v) is a basis of $\mathbb{R}^2$, so every x has a unique expansion

$$x - V_1 = \mu u + \nu v.$$

Taking determinants against v and against u and using bilinearity and $\det(u,u) = \det(v,v) = 0$,

$$\det(x - V_1, v) = \mu \det(u, v) = \mu\Delta, \qquad \det(u, x - V_1) = \nu \det(u, v) = \nu\Delta,$$

which is Cramer's rule in this instance; hence $\mu = \det(x - V_1, v)/\Delta$ and $\nu = \det(u, x - V_1)/\Delta$.

Now $x \in \operatorname{conv}\{V_1, V_2, V_3\}$ means $x = \lambda_1 V_1 + \lambda_2 V_2 + \lambda_3 V_3$ with $\lambda_i \geq 0$ summing to 1; substituting $\lambda_1 = 1 - \lambda_2 - \lambda_3$ gives $x - V_1 = \lambda_2 u + \lambda_3 v$, so by uniqueness $\mu = \lambda_2$ and $\nu = \lambda_3$. Therefore

$$x \in \operatorname{conv}\{V_1, V_2, V_3\} \iff \mu \geq 0,\ \nu \geq 0,\ \mu + \nu \leq 1.$$

The first two conditions are the first two displayed inequalities. For the third, write $x - V_2 = (x - V_1) - u = (\mu - 1)u + \nu v$ and $V_3 - V_2 = v - u$, so

$$\begin{aligned}\det(V_3 - V_2, x - V_2) &= \det\bigl(v - u,\ (\mu - 1)u + \nu v\bigr) \\ &= (\mu - 1)\det(v, u) - \nu \det(u, v) = \Delta\,(1 - \mu - \nu),\end{aligned}$$

using $\det(v, u) = -\det(u, v)$. Dividing by Δ turns $\mu + \nu \leq 1$ into the third inequality. □

The triangle is exhibited here both as a convex hull of three points and as an intersection of three half-planes. (Boundedness, needed for Definition 4.6, follows from the hull description and $\|\sum \lambda_i V_i\| \leq \max_i \|V_i\|$.)

Consider now the normals. Put $\varepsilon := \operatorname{sgn}\Delta \in \{-1, +1\}$, a sign local to this example (the chapter's free sign σ is not being fixed here). The first inequality reads $\varepsilon \det(u, x - V_1) \geq 0$, that is $\langle J_\varepsilon u,\ x - V_1\rangle \geq 0$, that is

$$\langle -J_\varepsilon u,\ x\rangle \ \leq\ \langle -J_\varepsilon u,\ V_1\rangle.$$

So the outward normal of the edge $[V_1, V_2]$, whose direction is u, is $-J_\varepsilon u = J_{-\varepsilon} u$, up to positive scaling: the outward normal of an edge is a quarter turn of the edge direction. The same computation applies to the other two edges. (When ε equals the sign σ frozen in later chapters, the outward normal is $-J_\sigma u$; when it does not, it is $+J_\sigma u$, since $J_{-\sigma} = -J_\sigma$.) Section 4.8 uses the same fact for the edges of a zonogon.

Chapter 13 uses the sign test of Example 4.11, applied to four points rather than three, when it discusses whether four centres can be in strictly convex position; Exercise 4.8 sets up that case.

4.3 Interiors and strict inequalities

Lemma 4.12 (no interior maximum)**.** *Let $X \subseteq \mathbb{R}^2$, let $n \neq 0$, and let $x \in \operatorname{int} X$. Then there is $y \in X$ with $\langle y, n\rangle > \langle x, n\rangle$.*

Proof. Choose $\rho > 0$ with $B(x, \rho) \subseteq X$ and put $y = x + \frac{\rho}{2\|n\|} n$. Then $\|y - x\| = \rho/2 < \rho$, so $y \in X$, and

$$\langle y, n\rangle = \langle x, n\rangle + \frac{\rho}{2\|n\|}\|n\|^2 = \langle x, n\rangle + \frac{\rho\|n\|}{2} > \langle x, n\rangle. \qquad \square$$

A nonzero linear functional therefore attains its maximum over a set only at points that are not interior. Convexity is not used; only the ball is. Chapter 5 quotes the lemma to say that an interior point satisfies $\langle x, n\rangle < h_X(n)$. Lemma 4.13 below is the quantitative form, naming the strict inequalities rather than asserting that one holds.

Lemma 4.13 (interior of a finite intersection of half-planes)**.** *Let $n_1, \dots, n_N$ be unit vectors, let $c_1, \dots, c_N \in \mathbb{R}$, and let $\Pi = \bigcap_{i=1}^N \{x : \langle x, n_i\rangle \leq c_i\}$. Then*

$$\operatorname{int}\Pi = \bigl\{x :\ \langle x, n_i\rangle < c_i \ \ \textit{for all } i\bigr\}.$$

Proof. ($\supseteq$) Suppose $\langle x, n_i\rangle < c_i$ for all i and put $\varepsilon = \min_i\bigl(c_i - \langle x, n_i\rangle\bigr)$, a minimum over a finite set, so $\varepsilon > 0$. If $\|y - x\| < \varepsilon$ then by Cauchy–Schwarz and $\|n_i\| = 1$,

$$\langle y, n_i\rangle = \langle x, n_i\rangle + \langle y - x, n_i\rangle \leq \langle x, n_i\rangle + \|y - x\| < \langle x, n_i\rangle + \varepsilon \leq c_i$$

for every i. Hence $B(x, \varepsilon) \subseteq \Pi$ and $x \in \operatorname{int}\Pi$.

($\subseteq$) Suppose $x \in \Pi$ but $\langle x, n_j\rangle = c_j$ for some j. For every $\rho > 0$ the point $x + \frac{\rho}{2} n_j$ lies in $B(x, \rho)$ and satisfies $\langle x + \frac{\rho}{2} n_j, n_j\rangle = c_j + \frac{\rho}{2} > c_j$, so it is not in Π; hence no ball about x lies in Π and $x \notin \operatorname{int}\Pi$. Points outside Π are not interior to it. □

Corollary 4.14. *For every frame $F = (e, f)$ and point P,*

$$\operatorname{int} Q(F,P) = \Big\{X : \ |\langle X-P, e\rangle| < \tfrac{1}{2} \ \ \textit{and} \ \ |\langle X-P, f\rangle| < \tfrac{1}{2}\Big\}.$$

Proof. $Q(F,P)$ is the intersection of the four half-planes $\langle X, \pm e\rangle \le \frac{1}{2} \pm \langle P, e\rangle$ and $\langle X, \pm f\rangle \le \frac{1}{2} \pm \langle P, f\rangle$, whose normals are unit vectors; apply Lemma 4.13 and recombine each pair of strict inequalities into one strict absolute-value inequality. □

This was proved directly in Chapter 1; it is recovered here as a special case of Lemma 4.13.

Proposition 4.15 (interior implies strict, in every direction)**.** *Let $F = (e, f)$ be a frame, $P \in \mathbb{R}^2$, and write*

$$w_F(n) := \frac{|\langle n, e\rangle| + |\langle n, f\rangle|}{2}$$

for a unit vector n. If $X \in \operatorname{int} Q(F,P)$ then

$$\langle X-P,\, n\rangle \ < \ w_F(n) \qquad \textit{for every unit vector } n.$$

Without the interiority hypothesis, $X \in Q(F,P)$ gives the non-strict inequality $\langle X-P, n\rangle \le w_F(n)$.

Proof. Since (e, f) is an orthonormal basis, write $X - P = a\,e + b\,f$ with $a = \langle X-P, e\rangle$, $b = \langle X-P, f\rangle$, and put $\alpha = \langle n, e\rangle$, $\beta = \langle n, f\rangle$; then $\alpha^2 + \beta^2 = \|n\|^2 = 1$, so α and β are not both 0. Bilinearity gives

$$\langle X-P, n\rangle = a\alpha + b\beta \le |a||\alpha| + |b||\beta|.$$

If $X \in Q(F,P)$ then $|a| \le \frac{1}{2}$ and $|b| \le \frac{1}{2}$, and the right-hand side is at most $\frac{1}{2}(|\alpha| + |\beta|) = w_F(n)$.

If $X \in \operatorname{int} Q(F,P)$ then, by Corollary 4.14, $|a| < \frac{1}{2}$ and $|b| < \frac{1}{2}$. Hence $|a||\alpha| \le \frac{1}{2}|\alpha|$ with strict inequality whenever $\alpha \ne 0$, and likewise for β. As at least one of α, β is nonzero, at least one of the two estimates is strict, and the sum is strictly less than $w_F(n)$. □

Chapter 5 calls w_F the *projected half-width* and develops its properties. It is even, $w_F(-n) = w_F(n)$, and is unchanged if the frame F is replaced by any other ordered pair of perpendicular axes of F, since only the absolute values $|\langle n, \pm e\rangle|$, $|\langle n, \pm f\rangle|$ enter.

Corollary 4.16 (a single direction certifies disjointness)**.** *Let F, G be frames, $P, Q \in \mathbb{R}^2$, and let n be a unit vector.*

(i) If some X lies in $\operatorname{int} Q(F,P) \cap \operatorname{int} Q(G,Q)$ then $\langle Q-P, n\rangle < w_F(n) + w_G(n)$.

(ii) Consequently, if $\langle Q-P, n\rangle \ge w_F(n) + w_G(n)$ for a single unit vector n, then $\operatorname{int} Q(F,P)$ and $\operatorname{int} Q(G,Q)$ are disjoint.

Proof. (i) Apply Proposition 4.15 twice: $\langle X-P, n\rangle < w_F(n)$ and $\langle Q-X, n\rangle < w_G(n)$, the latter because $X \in \operatorname{int} Q(G,Q)$ gives $\langle X-Q, -n\rangle < w_G(-n) = w_G(n)$. Adding, $\langle Q-P, n\rangle < w_F(n) + w_G(n)$.

(ii) The contrapositive of (i). □

Chapter 5 calls the inequality in (ii) an *oriented separating branch*; its sufficiency for disjoint interiors is the two lines just given.

Lemma 4.17 (radial interior lemma)**.** *Let $X \subseteq \mathbb{R}^2$ be convex with $0 \in \operatorname{int} X$. If $x \in X$ and $0 \le \mu < 1$ then $\mu x \in \operatorname{int} X$.*

Proof. Choose $\rho > 0$ with $B(0,\rho) \subseteq X$, and let z satisfy $\|z - \mu x\| < (1-\mu)\rho$. Put $y = (z - \mu x)/(1-\mu)$, so $\|y\| < \rho$ and hence $y \in X$. Then

$$z = \mu x + (1-\mu) y$$

is a convex combination of the points $x, y \in X$, so $z \in X$ by convexity. Therefore $B(\mu x, (1-\mu)\rho) \subseteq X$ and $\mu x \in \operatorname{int} X$. □

4.4 The unit square as an intersection of two slabs

Recall from Chapter 1 that a *frame* is an ordered orthonormal pair $F = (e, f)$, with *axes* the four unit vectors $\pm e, \pm f$, and that

$$Q(F,P) = \Big\{X : \ |\langle X-P, e\rangle| \le \tfrac{1}{2} \ \text{ and } \ |\langle X-P, f\rangle| \le \tfrac{1}{2}\Big\} \qquad (P \in \mathbb{R}^2).$$

The definition is unchanged; what follows is a reading of it. Each of the two conditions is a slab, so $Q(F,P)$ is an intersection of two slabs, hence of the four half-planes with unit normals $\pm e, \pm f$ at levels $\frac{1}{2} \pm \langle P, e\rangle$, $\frac{1}{2} \pm \langle P, f\rangle$. It is bounded: by orthonormality

$$\|X-P\|^2 = \langle X-P, e\rangle^2 + \langle X-P, f\rangle^2 \le \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{2},$$

so $\|X-P\| \le 1/\sqrt{2}$ on $Q(F,P)$. Hence $Q(F,P)$ is a convex polygon in the sense of Definition 4.6, and every general statement of Section 4.1 applies to it. Later chapters work with $Q(F,P)$ in this form, through the frame axes and inner products alone.

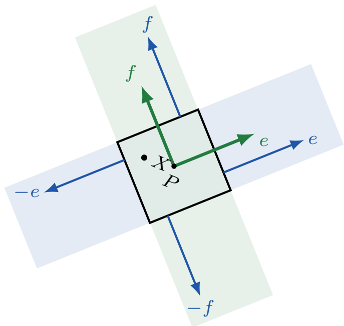


Figure 4.2: $Q(F,P)$ as the intersection of two slabs, one per axis direction. The frame axes are green, the four outward unit normals $\pm e, \pm f$ blue; all six arrows are drawn at their true length 1, so the green axes protrude beyond the square, whose side is 1. An interior point X satisfies $|\langle X-P, e\rangle| < \frac{1}{2}$ and $|\langle X-P, f\rangle| < \frac{1}{2}$, and by Proposition 4.15 a strict inequality in every direction, not only these four.

Proposition 4.18. *Let $F = (e, f)$ be a frame and $P \in \mathbb{R}^2$. Call a subset of $\mathbb{R}^2$ a* closed unit square *if it is the image of $[-\frac{1}{2}, \frac{1}{2}]^2$ under a map $x \mapsto Mx + b$ with M a 2×2 orthogonal matrix and $b \in \mathbb{R}^2$.*

(i) $Q(F,P)$ is the image of $[-\frac{1}{2}, \frac{1}{2}]^2$ under the affine isometry $\Phi(x_1, x_2) = P + x_1 e + x_2 f$; it is a closed unit square, of side 1 and area 1; and conversely every closed unit square is $Q(F,P)$ for some frame F and some P.

(ii) $Q(F,P) = \operatorname{conv}\big\{P \pm \frac{1}{2}e \pm \frac{1}{2}f\big\}$, the hull of the four points $P \pm \frac{1}{2}e \pm \frac{1}{2}f$, which are its vertices in the sense of Definition 4.6.

(iii) $Q(F,P)$ depends on F only through the set $\{\pm e, \pm f\}$ of its axes. In particular $Q\big((e,f),P\big) = Q\big((f,-e),P\big) = Q\big((-e,f),P\big)$, and there are exactly eight ordered pairs of perpendicular axes of F.

Proof. (i) Since (e, f) is an orthonormal basis, every X is uniquely $X = P + x_1 e + x_2 f$ with $x_1 = \langle X-P, e\rangle$, $x_2 = \langle X-P, f\rangle$, and

$$\|\Phi(x) - \Phi(y)\|^2 = \|(x_1 - y_1)e + (x_2 - y_2)f\|^2 = (x_1 - y_1)^2 + (x_2 - y_2)^2$$

by orthonormality, so Φ is an affine isometry of $\mathbb{R}^2$ onto itself, and $\Phi(x) = Mx + P$ with M the matrix whose columns are e and f, which is orthogonal because (e, f) is orthonormal. The defining inequalities of $Q(F,P)$ are $|x_1| \le \frac{1}{2}$, $|x_2| \le \frac{1}{2}$, so $Q(F,P) = \Phi([-\frac{1}{2}, \frac{1}{2}]^2)$; side, area and closedness are preserved by isometries. Conversely, let $S = \{Mx + b : x \in [-\frac{1}{2}, \frac{1}{2}]^2\}$ with M orthogonal. Put $e = Me_1$ and $f = Me_2$, the columns of M; orthogonality of M says exactly that (e, f) is orthonormal, i.e. a frame, and $Mx + b = b + x_1 e + x_2 f$, so $S = Q((e, f), b)$.

(ii) By (i) and Lemma 4.5(iii), applying the affine map Φ to both sides reduces the claim to $[-\frac{1}{2},\frac{1}{2}]^2 = \mathrm{conv}\{(\pm\frac{1}{2},\pm\frac{1}{2})\}$, since affine maps commute with convex combinations and hence with conv. The inclusion $\supseteq$ holds because $[-\frac{1}{2},\frac{1}{2}]^2$ is convex (a slab intersection) and contains the four corners, and Lemma 4.9 applies. For $\subseteq$, let $|a| \le \frac{1}{2}$, $|b| \le \frac{1}{2}$ and put $p = a + \frac{1}{2} \in [0,1]$, $q = b + \frac{1}{2} \in [0,1]$. Then

$$(a,b) = pq\left(\tfrac{1}{2},\tfrac{1}{2}\right) + p(1-q)\left(\tfrac{1}{2},-\tfrac{1}{2}\right) + (1-p)q\left(-\tfrac{1}{2},\tfrac{1}{2}\right) + (1-p)(1-q)\left(-\tfrac{1}{2},-\tfrac{1}{2}\right),$$

as one checks by expanding: the first coordinate is $\frac{1}{2}\big(pq + p(1-q) - (1-p)q - (1-p)(1-q)\big) = \frac{1}{2}\big(p-(1-p)\big) = a$, and similarly for the second. The four coefficients are non-negative and sum to $\big(p+(1-p)\big)\big(q+(1-q)\big) = 1$. Finally, each of the four points is a vertex: $P + \frac{1}{2}e + \frac{1}{2}f$ lies on the facet with normal e and on the facet with normal f, and e, f are linearly independent; likewise for the other three sign choices.

(iii) The two defining conditions involve e and f only through $|\langle X-P, e\rangle|$ and $|\langle X-P, f\rangle|$, so they are unchanged by replacing e by $-e$, or f by $-f$, or by interchanging the two conditions, i.e. by swapping e and f. Every ordered pair of perpendicular axes of F is obtained from (e,f) by such moves, and there are $4 \times 2 = 8$ such ordered pairs: four choices of first entry, and then two perpendicular to it. ☐

Part (iii) says that at least eight frames present the same square. That exactly eight do, equivalently that $Q(F,P)$ determines P and the axis set, is the next lemma; it does not follow from (iii), and the frame circle of Chapter 5 needs it.

Lemma 4.19 (a square determines its centre and its axes)**.** *Let $F = (e,f)$ and $F' = (e',f')$ be frames and $P, P' \in \mathbb{R}^2$, and suppose $Q(F,P) = Q(F',P')$. Then $P = P'$ and $\{\pm e, \pm f\} = \{\pm e', \pm f'\}$. Consequently the frames F' with $Q(F',P') = Q(F,P)$ are exactly the eight ordered pairs of perpendicular axes of F, all with $P' = P$.*

Proof. Write $S = Q(F,P)$ and let $V = \{P \pm \frac{1}{2}e \pm \frac{1}{2}f\}$ be the four points of Proposition 4.18(ii). We recover P and $\{\pm e, \pm f\}$ from S alone.

Step 1: the pairs of points of S at distance $\sqrt{2}$ are exactly the two pairs of opposite points of V, and each such pair has midpoint P. Let $X, Y \in S$ and write $X - P = ae + bf$, $Y - P = a'e + b'f$, so that $|a|, |b|, |a'|, |b'| \le \frac{1}{2}$. By orthonormality,

$$\|X-Y\|^2 = (a-a')^2 + (b-b')^2 \le 1 + 1 = 2,$$

since $|a-a'| \le 1$ and $|b-b'| \le 1$. Equality forces $|a-a'| = |b-b'| = 1$, hence $\{a,a'\} = \{\frac{1}{2},-\frac{1}{2}\}$ and $\{b,b'\} = \{\frac{1}{2},-\frac{1}{2}\}$; that is, $X, Y \in V$ with $X - P = -(Y-P)$. Conversely each such pair does realise the distance $\sqrt{2}$. For such a pair, $X + Y = 2P$, so $P = \frac{1}{2}(X+Y)$.

Step 2: P and V are determined by S. By Step 1, P is the midpoint of any pair of points of S at distance $\sqrt{2}$, and such a pair exists, so P is determined by S. Then V is determined too: it is the set of points of S belonging to some such pair. (Equivalently, $V = \{X \in S : \|X-P\| = 1/\sqrt{2}\}$, the points of S farthest from P.)

Step 3: the axis set is determined by V. The differences $X - Y$ with $X, Y \in V$ are 0, the four vectors $\pm e, \pm f$ (of norm 1) and the four vectors $\pm e \pm f$ (of norm $\sqrt{2}$). Hence

$$\{\pm e, \pm f\} = \big\{X - Y \;:\; X, Y \in V,\ \|X-Y\| = 1\big\},$$

a set determined by V, hence by S.

Applying Steps 1–3 to the two presentations of the same set gives $P = P'$ and $\{\pm e, \pm f\} = \{\pm e', \pm f'\}$. Since (e',f') is orthonormal, it is an ordered pair of perpendicular axes of F; and by Proposition 4.18(iii) every one of those eight ordered pairs does present S. ☐

This is why Chapter 5 may regard a frame as a point of the circle $\mathbb{R}/(\pi/2)\mathbb{Z}$: by Lemma 4.19 the geometric object determines the frame exactly modulo the eight-element group of quarter turns and sign changes, and the quotient of the circle of directions by that group is a circle of circumference $\pi/2$.

Example 4.20 (half-widths of a square)**.** Take F to be the standard frame, $e = (1, 0)$, $f = (0, 1)$, and $n = (\cos\theta, \sin\theta)$. Then

$$w_F(n) = \frac{|\cos\theta| + |\sin\theta|}{2}.$$

By the unit-vector companion of Chapter 2 ($1 \le |a| + |b| \le \sqrt{2}$ when $a^2 + b^2 = 1$),

$$\tfrac{1}{2} \ \le\ w_F(n) \ \le\ \tfrac{1}{\sqrt{2}},$$

with the minimum attained exactly when n is an axis of F (one of $\cos\theta$, $\sin\theta$ vanishes) and the maximum exactly when n is one of the four diagonal directions $\theta \in \{\pi/4, 3\pi/4, 5\pi/4, 7\pi/4\}$.

By Proposition 4.15 and its non-strict half, the set of values $\langle X, n\rangle$ for $X \in Q(F, 0)$ lies in $[-w_F(n), w_F(n)]$, and both endpoints are attained (at $\pm\frac{1}{2}(\operatorname{sgn}\langle n, e\rangle\, e + \operatorname{sgn}\langle n, f\rangle\, f)$). So $2w_F(n)$ is the *extent* (largest value minus smallest) of the *shadow* of the square on a line in direction n; that the shadow is the whole interval $[-w_F(n), w_F(n)]$, rather than merely contained in it with both endpoints attained, is Exercise 4.3(c). A unit square therefore casts a shadow of extent between 1 and $\sqrt{2}$ in every orientation. The lower bound $w_F \ge \frac{1}{2}$ is the inequality $|a| + |b| \ge 1$, and is quoted in Chapter 5.

4.5 Minkowski sums

Definition 4.21. For $X, Y \subseteq \mathbb{R}^2$, $x \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$,

$$X + Y = \{a + b : a \in X,\ b \in Y\}, \quad X - Y = \{a - b : a \in X,\ b \in Y\},$$

$$x + X = \{x + a : a \in X\}, \quad \lambda X = \{\lambda a : a \in X\}.$$

The set $X + Y$ is the *Minkowski sum.*

Throughout, $X - Y$ is the Minkowski difference $\{a - b\}$, not the set difference $\{a \in X : a \notin Y\}$.

Lemma 4.22. *Let $X, Y, Y' \subseteq \mathbb{R}^2$.*
(i) $X + Y = Y + X$ and $(X + Y) + Y' = X + (Y + Y')$, and $(x + X) + (y + Y) = (x + y) + (X + Y)$.
(ii) If X and Y are convex then so is $X + Y$.
(iii) If X is convex then $X + X = 2X$.

Proof. (i) Immediate from commutativity and associativity of vector addition.

(ii) Let $a + b$, $a' + b'$ lie in $X + Y$ and $\lambda \in [0, 1]$. Then

$$(1 - \lambda)(a + b) + \lambda(a' + b') = \big[(1 - \lambda)a + \lambda a'\big] + \big[(1 - \lambda)b + \lambda b'\big] \in X + Y.$$

(iii) $2X \subseteq X + X$ since $2a = a + a$. Conversely if $a, b \in X$ then $\frac{1}{2}(a + b) \in X$ by convexity, so $a + b \in 2X$. □

Definition 4.23 (central symmetry)**.** $X \subseteq \mathbb{R}^2$ is *centrally symmetric* (about 0) if $X = -X$.

Lemma 4.24. *If $Y = -Y$ then $X + Y = X - Y$ for every X. Moreover $Q(F, 0) = -Q(F, 0)$ for every frame F, and likewise $\operatorname{int} Q(F, 0) = -\operatorname{int} Q(F, 0)$.*

Proof. $X - Y = X + (-Y) = X + Y$. The defining conditions of $Q(F, 0)$ are $|\langle X, e\rangle| \le \frac{1}{2}$ and $|\langle X, f\rangle| \le \frac{1}{2}$, which are unchanged when X is replaced by $-X$; the same for the strict versions, using Corollary 4.14. □

Lemma 4.25 (interior of a sum)**.** *If $x \in \operatorname{int} X$ and $y \in Y$ then $x + y \in \operatorname{int}(X + Y)$.*

Proof. Choose $\rho > 0$ with $B(x, \rho) \subseteq X$. Then $B(x + y, \rho) = B(x, \rho) + y \subseteq X + Y$. □

Example 4.26 (a square is a sum of two segments)**.** For a frame $F = (e, f)$,

$$Q_F := Q(F, 0) = \big[-\tfrac{1}{2}e, \tfrac{1}{2}e\big] + \big[-\tfrac{1}{2}f, \tfrac{1}{2}f\big] = \big\{ae + bf : \ |a| \le \tfrac{1}{2}, \ |b| \le \tfrac{1}{2}\big\}.$$

Indeed a point of the middle set is $ae + bf$ with $|a|, |b| \le \frac{1}{2}$, and by orthonormality $a = \langle X, e\rangle$ and $b = \langle X, f\rangle$ for $X = ae + bf$, so the middle and right sets coincide, and the right set is Q_F by definition.

Consequently, for two frames $F = (e, f)$ and $G = (e', f')$,

$$Q_F + Q_G = \big[-\tfrac{1}{2}e, \tfrac{1}{2}e\big] + \big[-\tfrac{1}{2}f, \tfrac{1}{2}f\big] + \big[-\tfrac{1}{2}e', \tfrac{1}{2}e'\big] + \big[-\tfrac{1}{2}f', \tfrac{1}{2}f'\big]$$

is a Minkowski sum of four segments, whose directions are the four axes e, f, e', f'. Section 4.8 studies this set.

Example 4.27 (the rounded square, and the centre-separation lemma)**.** The set $Q_F + \bar{B}(0, \frac{1}{2})$ is the unit square with a half-disc glued along each edge and a quarter-disc at each corner: a "rounded square". It is the obstacle for a square-and-disc pair in the sense of the next section (Exercise 4.6). Note also that $\bar{B}(0, \frac{1}{2}) + \bar{B}(0, \frac{1}{2}) = \bar{B}(0, 1)$, and likewise for open balls: the inclusion $\subseteq$ is the triangle inequality, and $\supseteq$ holds because $x = \frac{x}{2} + \frac{x}{2}$.

The centre-separation lemma of Chapter 1 reappears. That chapter showed $B(P, \frac{1}{2}) \subseteq \operatorname{int} Q(F, P)$. Suppose two unit squares $Q(F, P)$ and $Q(G, Q)$ have $\|Q - P\| < 1$, and let $M = \frac{1}{2}(P + Q)$. Then $\|M - P\| = \frac{1}{2}\|Q - P\| < \frac{1}{2}$, so $M \in B(P, \frac{1}{2}) \subseteq \operatorname{int} Q(F, P)$, and symmetrically $M \in \operatorname{int} Q(G, Q)$; the two interiors meet. Contrapositively, disjoint interiors force $\|Q - P\| \ge 1$. In the language of this section, the obstacle for the two balls of radius $\frac{1}{2}$ is $\bar{B}(0, \frac{1}{2}) + \bar{B}(0, \frac{1}{2}) = \bar{B}(0, 1)$.

4.6 The obstacle set

Proposition 4.28 (when two translates meet)**.** *For all sets $X, Y \subseteq \mathbb{R}^2$ and all points P, Q,*

$$(P + X) \cap (Q + Y) \ne \emptyset \quad \Longleftrightarrow \quad Q - P \in X - Y.$$

Proof. A common point is $P + a = Q + b$ with $a \in X$, $b \in Y$, and this equation is equivalent to $Q - P = a - b$. So a common point exists if and only if $Q - P$ is of the form $a - b$. □

Theorem 4.29 (the obstacle set)**.** *Let F, G be frames, let $Q_F = Q(F, 0)$, $Q_G = Q(G, 0)$, and put*

$$Z := Q_F + Q_G.$$

Then for all centres P, Q,

$$\operatorname{int} Q(F, P) \cap \operatorname{int} Q(G, Q) \ne \emptyset \quad \Longleftrightarrow \quad Q - P \in \operatorname{int} Z.$$

Proof. First, $Q(F, P) = P + Q_F$ and $\operatorname{int} Q(F, P) = P + \operatorname{int} Q_F$: the defining conditions depend on X only through $X - P$ (by Corollary 4.14 for the interior); the same for G and Q. Second, by Lemma 4.24, $Q_G = -Q_G$ and hence

$$Z = Q_F + Q_G = Q_F - Q_G.$$

($\Rightarrow$) Let X lie in both interiors. Put $a = X - P \in \operatorname{int} Q_F$ and $b = X - Q \in \operatorname{int} Q_G$. Then $Q - P = a - b$, and $-b \in \operatorname{int} Q_G$ by central symmetry, so $Q - P = a + (-b)$ with $a \in \operatorname{int} Q_F$ and $-b \in Q_G$. By Lemma 4.25, $Q - P \in \operatorname{int}(Q_F + Q_G) = \operatorname{int} Z$.

($\Leftarrow$) Let $x = Q - P \in \operatorname{int} Z$ and choose $\delta_0 > 0$ with $B(x, \delta_0) \subseteq Z$. Choose $\delta > 0$ with $\delta\|x\| < \delta_0$ (any $\delta > 0$ will do if $x = 0$). Then $(1 + \delta)x$ lies in $B(x, \delta_0)$, hence in Z, so

$$(1 + \delta)x = a + b, \qquad a \in Q_F, \ b \in Q_G,$$

and therefore

$$x = \mu a + \mu b, \qquad \mu := \frac{1}{1 + \delta} \in (0, 1).$$

Both Q_F and Q_G are convex and contain 0 in their interiors (by Corollary 4.14, since $\langle 0, e\rangle = \langle 0, f\rangle = 0$), so Lemma 4.17 gives

$$a' := \mu a \in \operatorname{int} Q_F, \qquad b' := \mu b \in \operatorname{int} Q_G, \qquad x = a' + b'.$$

Now set $X := P + a'$. Then $X - P = a' \in \operatorname{int} Q_F$, so $X \in \operatorname{int} Q(F, P)$; and

$$X - Q = P + a' - Q = a' - x = -b',$$

which lies in $\operatorname{int} Q_G$ by central symmetry, so $X \in \operatorname{int} Q(G, Q)$. The two interiors therefore meet. $\square$

The set Z depends only on the two frames, not on the centres. Chapter 9 uses this: with the frames held constant, each pair of squares has one fixed obstacle, and only the point $Q - P$ varies.

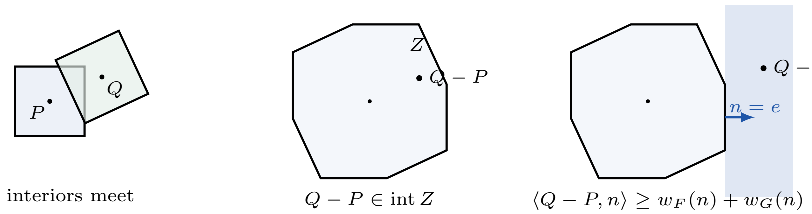


Figure 4.3: The obstacle reformulation. Left: two unit squares whose interiors meet. Centre: the same information carried by the single point $Q - P$ inside the fixed octagon $Z = Q_F + Q_G$. Right: when the interiors are disjoint the point leaves Z, and by Theorem 4.39 it violates one of eight explicit inequalities, whose shaded half-plane is drawn.

4.7 Support functions

Definition 4.30. For a nonempty bounded set $X \subseteq \mathbb{R}^2$ and $n \in \mathbb{R}^2$, the *support function* of X is

$$h_X(n) = \sup_{x \in X} \langle x, n\rangle.$$

The supremum is finite because X is bounded and $|\langle x, n\rangle| \le \|x\|\|n\|$. It need not be attained: for the open ball, $h_{B(0,\rho)}(n) = \rho\|n\|$ is a supremum and no point of $B(0, \rho)$ achieves it. This is harmless, since every statement below, additivity included, is proved for suprema. For a segment, for a square, for a zonogon and for the closed ball $\bar{B}(0, \rho)$ a maximiser is exhibited, and we then write "max".

Proposition 4.31. *For every frame $F = (e, f)$ and every unit vector n,*

$$h_{Q_F}(n) = w_F(n) = \frac{|\langle n, e\rangle| + |\langle n, f\rangle|}{2},$$

and the maximum is attained at the vertex

$$x^* = \tfrac{1}{2}\big(\operatorname{sgn}\langle n, e\rangle\, e + \operatorname{sgn}\langle n, f\rangle\, f\big),$$

with either sign chosen when a coefficient vanishes.

Proof. By Proposition 4.18(ii) with $P = 0$, Q_F is the convex hull of the four points $\frac{1}{2}(\varepsilon_1 e + \varepsilon_2 f)$ with $\varepsilon_1, \varepsilon_2 \in \{-1, +1\}$. So Lemma 4.10 applies and replaces the supremum over Q_F by a maximum over those four points:

$$h_{Q_F}(n) = \max_{\varepsilon_1, \varepsilon_2 \in \{-1,+1\}} \tfrac{1}{2}\big(\varepsilon_1\langle e, n\rangle + \varepsilon_2\langle f, n\rangle\big) = \tfrac{1}{2}\big(|\langle n, e\rangle| + |\langle n, f\rangle|\big) = w_F(n),$$

since the two signs are chosen independently and $\max_{\varepsilon=\pm1} \varepsilon a = |a|$. The maximum is attained at the point with ε_i the sign of the corresponding inner product, which is x^*; and x^* is a vertex of Q_F by Proposition 4.18(ii). $\square$

The projected half-width of Chapter 5 is thus a support function.

Proposition 4.32 (calculus of support functions)**.** *Let $X, Y \subseteq \mathbb{R}^2$ be nonempty and bounded, $x \in \mathbb{R}^2$, $\lambda \geq 0$, $n \in \mathbb{R}^2$. Then*

(i) $h_{X+Y} = h_X + h_Y$*;*
(ii) $h_{x+X}(n) = \langle x, n\rangle + h_X(n)$*;*
(iii) $h_{\lambda X} = \lambda h_X$ *and* $h_X(\lambda n) = \lambda h_X(n)$*;*
(iv) $h_{-X}(n) = h_X(-n)$*; in particular if* $X = -X$ *then* h_X *is even;*
(v) if $X \subseteq Y$ *then* $h_X \leq h_Y$*.*

Proof. (i) $\sup_{a\in X, b\in Y}\big(\langle a, n\rangle + \langle b, n\rangle\big) = \sup_{a\in X}\langle a, n\rangle + \sup_{b\in Y}\langle b, n\rangle$, because the two summands range independently: each term of the left supremum is at most the right-hand side, and conversely $\langle a, n\rangle + \langle b, n\rangle$ can be made to exceed $h_X(n) + h_Y(n) - \varepsilon$ by choosing a and b separately within $\varepsilon/2$ of their suprema.

(ii)–(v) Each follows from the definition: for (ii), $\langle x + a, n\rangle = \langle x, n\rangle + \langle a, n\rangle$; for (iii), $\langle \lambda a, n\rangle = \lambda\langle a, n\rangle$ and $\langle a, \lambda n\rangle = \lambda\langle a, n\rangle$, with $\lambda \geq 0$ so that the supremum is unaffected by the positive scaling; for (iv), $\langle -a, n\rangle = \langle a, -n\rangle$; for (v), a supremum over a larger set is at least as large. □

Corollary 4.33. *With $Z = Q_F + Q_G$ as in Theorem 4.29, for every unit vector n,*

$$h_Z(n) = w_F(n) + w_G(n).$$

Proof. Propositions 4.31 and 4.32(i). □

The quantity on the right is the *threshold* $H(F, G; n)$ of Chapter 5: the sum of two one-dimensional shadow half-lengths is the support function of the single obstacle Z. Thresholds add because support functions do, Proposition 4.32(i).

Lemma 4.34 (a set lies in its supporting half-planes)**.** *For every nonempty bounded X and every n, $X \subseteq \{x : \langle x, n\rangle \leq h_X(n)\}$. Hence, for any set N of directions,*

$$X \subseteq \bigcap_{n\in N} \{x : \langle x, n\rangle \leq h_X(n)\}.$$

Proof. By definition $\langle x, n\rangle \leq h_X(n)$ for every $x \in X$. □

Remark 4.35 (the reverse inclusion). The reverse inclusion is not available here. For a finite set N of directions it is false in general: take X a disc and $N = \{e, -e\}$, when the right-hand side is an unbounded slab. For the set of all directions the reverse inclusion is true for closed convex X, but it is a separation theorem, proved in this book only in Chapter 8, by way of a nearest-point projection argument that needs the extreme value theorem of Chapter 7. For the obstacle Z, the reverse inclusion is proved directly in the next section, from the eight axis normals.

4.8 Zonogons, and the octagon theorem

Definition 4.36. Let $g_1, \ldots, g_k \in \mathbb{R}^2$ be nonzero. The *zonogon* they generate is

$$\mathrm{Zon}(g_1, \ldots, g_k) = \sum_{j=1}^{k}\big[-\tfrac12 g_j, \tfrac12 g_j\big] = \Big\{\sum_{j=1}^{k}\lambda_j g_j \;:\; |\lambda_j| \leq \tfrac12 \text{ for all } j\Big\}.$$

Lemma 4.37. $\mathrm{Zon}(g_1, \ldots, g_k)$ *is convex, bounded, contains* 0*, and is centrally symmetric. Moreover, for every n,*

$$h_{\mathrm{Zon}(g_1,\ldots,g_k)}(n) = \frac12\sum_{j=1}^{k}\big|\langle n, g_j\rangle\big|,$$

attained at the point with $\lambda_j = \frac12 \operatorname{sgn}\langle n, g_j\rangle$ for each j (either sign when $\langle n, g_j\rangle = 0$).

Proof. Each segment $[-\frac{1}{2}g_j, \frac{1}{2}g_j]$ is convex (it is the hull of two points), is bounded, contains 0, and equals its own negative. Sums of convex sets are convex by Lemma 4.22(ii), and sums of bounded sets are bounded; and $-\sum_j \lambda_j g_j = \sum_j (-\lambda_j) g_j$ shows central symmetry. For the support function, a segment is the hull of its two endpoints, $[-\frac{1}{2}g, \frac{1}{2}g] = \operatorname{conv}\{-\frac{1}{2}g, \frac{1}{2}g\}$, so Lemma 4.10 gives

$$h_{[-\frac{1}{2}g,\,\frac{1}{2}g]}(n) = \max\big\{\tfrac{1}{2}\langle g, n\rangle,\ -\tfrac{1}{2}\langle g, n\rangle\big\} = \tfrac{1}{2}|\langle n, g\rangle|,$$

attained at the endpoint $\frac{1}{2}\operatorname{sgn}\langle n, g\rangle\, g$; now apply additivity, Proposition 4.32(i), and note that the maximisers add. □

Proposition 4.38. *Let $F = (e, f)$ and $G = (e', f')$ be frames. Then $Z = Q_F + Q_G = \operatorname{Zon}(e, f, e', f')$.*

Proof. Example 4.26 and associativity. □

Why the normals will be frame axes. Consider one generator g and its segment $[-\frac{1}{2}g, \frac{1}{2}g]$. The two lines supporting it have outward normals $\pm J_\sigma g/\|g\|$, the quarter turns of the direction, as for a triangle edge in Example 4.11. So the candidate facet normals of a zonogon are the quarter turns of its generators. The generators of Z are e, f, e', f', and $J_\sigma e = \pm f$, $J_\sigma f = \mp e$, $J_\sigma e' = \pm f'$, $J_\sigma f' = \mp e'$, by Chapter 2. Hence the quarter turns of the four generators of Z are again axes of F and of G.

Theorem 4.39 (octagon theorem)**.** *Let $F = (e, f)$ and $G = (e', f')$ be frames, let $Z = Q_F + Q_G$, and let*

$$N = \{\pm e, \pm f, \pm e', \pm f'\}$$

be the set of the eight axes (a set of four or eight distinct unit vectors). Then

$$Z = \big\{x :\ \langle x, n\rangle \le w_F(n) + w_G(n) \quad \textit{for all } n \in N\big\},$$

$$\operatorname{int} Z = \big\{x :\ \langle x, n\rangle < w_F(n) + w_G(n) \quad \textit{for all } n \in N\big\}.$$

Moreover the axes may be relabelled so that $e' = ce + sf$ and $f' = -se + cf$ with $c = \cos t \ge 0$, $s = \sin t \ge 0$ and $t \in [0, \pi/2]$, and then all eight numbers $w_F(n) + w_G(n)$, $n \in N$, are equal to the single number

$$H := \frac{1 + c + s}{2} = \frac{1 + \cos t + \sin t}{2} \ \ge\ 1.$$

The relabelling in Step 0 below does not determine t uniquely: it selects the axis of G lying in the first quadrant of the coordinates of (e, f), so interchanging the roles of e and f replaces t by $\pi/2 - t$. Since $\cos(\pi/2 - t) + \sin(\pi/2 - t) = \sin t + \cos t$, H is unchanged; equivalently, H depends on the pair of frames only through $\min\{t, \pi/2 - t\} \in [0, \pi/4]$. That number is the frame distance $d_4(F, G)$ of Chapter 5, and the function $t \mapsto (1 + \cos t + \sin t)/2$ is its $H(t)$.

Proof. Step 0: normalisation. By Proposition 4.18(iii), Q_G is unchanged if (e', f') is replaced by any ordered pair of perpendicular axes of G, and w_G is likewise unchanged (it involves the axes only through absolute values of inner products). The four axes of G are, in the orthonormal coordinates of (e, f), the four vectors (a, b), $(-b, a)$, $(-a, -b)$, $(b, -a)$ for some $a^2 + b^2 = 1$; exactly one of these four lies in the closed first quadrant unless $ab = 0$, in which case two do. Choose e' to be one lying there, so that

$$c := \langle e', e\rangle \ge 0, \qquad s := \langle e', f\rangle \ge 0, \qquad c^2 + s^2 = 1,$$

and write $c = \cos t$, $s = \sin t$ with $t \in [0, \pi/2]$. The two unit vectors orthogonal to e' are $\pm(-se + cf)$, and both are axes of G; take $f' = -se + cf$. Then (e', f') is a frame with the axes of G, so $Q((e', f'), 0) = Q_G$, and the set N is unchanged. From now on we work in the coordinates of the orthonormal basis (e, f):

$$e = (1, 0), \quad f = (0, 1), \quad e' = (c, s), \quad f' = (-s, c).$$

Step 1: all eight thresholds are equal. Directly from the definition of w,

$$\begin{aligned} w_F(e) &= \tfrac{1}{2}(|1| + |0|) = \tfrac{1}{2}, & w_F(f) &= \tfrac{1}{2}, \\ w_G(e) &= \tfrac{1}{2}\big(|\langle e, e'\rangle| + |\langle e, f'\rangle|\big) = \tfrac{c+s}{2}, & w_G(f) &= \tfrac{1}{2}(s+c) = \tfrac{c+s}{2}, \\ w_F(e') &= \tfrac{1}{2}(c+s), & w_F(f') &= \tfrac{1}{2}(s+c), \\ w_G(e') &= \tfrac{1}{2}, & w_G(f') &= \tfrac{1}{2}, \end{aligned}$$

using $c, s \geq 0$. Since w_F and w_G are even, the same values occur at the four negatives $-e$, $-f$, $-e'$, $-f'$. Hence for every $n \in N$,

$$h_Z(n) = w_F(n) + w_G(n) = \frac{1+c+s}{2} = H,$$

the first equality by Corollary 4.33. Note also that $(c+s)^2 = 1 + 2cs \geq 1$, so $c + s \geq 1$ and $H \geq 1$. Write

$$W := \big\{x : \ \langle x, n\rangle \leq H \ \text{ for all } n \in N\big\};$$

the theorem's first display is the assertion $Z = W$.

Step 2: $Z \subseteq W$. Immediate from Lemma 4.34 and Step 1.

Step 3: the degenerate case $cs = 0$. If $s = 0$ then $c = 1$, $e' = e$, $f' = f$; if $c = 0$ then $s = 1$, $e' = f$, $f' = -e$. In both cases the axes of G are the axes of F, so $Q_G = Q_F$ by Proposition 4.18(iii), and $N = \{\pm e, \pm f\}$. Then $H = 1$, and $Z = Q_F + Q_F = 2Q_F$ by Lemma 4.22(iii), that is,

$$Z = \big\{x : |\langle x, e\rangle| \leq 1, \ |\langle x, f\rangle| \leq 1\big\} = W.$$

So the theorem holds when $cs = 0$. For the rest of the proof assume $c > 0$ and $s > 0$.

Step 4: the facet slice at $n = e$. We compute $W \cap \{x : \langle x, e\rangle = H\}$, that is, $W \cap \{x_1 = H\}$. Put

$$\alpha := \frac{1+c-s}{2}, \qquad \beta := \frac{1-c+s}{2},$$

two numbers local to this proof. For a point $x = (H, x_2)$, the eight constraints read: $\langle x, e\rangle = H \leq H$ (equality) and $\langle x, -e\rangle = -H \leq H$ (true, as $H > 0$); $|x_2| \leq H$ from $\pm f$; and, from $\pm e'$ and $\pm f'$,

$$|cH + sx_2| \leq H, \qquad |-sH + cx_2| \leq H,$$

which, since $s > 0$ and $c > 0$, are the four bounds

$$\begin{aligned} x_2 &\leq \frac{H(1-c)}{s}, & x_2 &\geq -\frac{H(1+c)}{s}, \\ x_2 &\leq \frac{H(1+s)}{c}, & x_2 &\geq -\frac{H(1-s)}{c}. \end{aligned}$$

Now use the two exact identities

$$(1+c+s)(1-c) = s\,(1+s-c), \qquad (1+c+s)(1-s) = c\,(1+c-s),$$

each of which follows from $c^2 + s^2 = 1$: for the first, $(1+c+s)(1-c) = 1 - c^2 + s - sc = s^2 + s - sc = s(1+s-c)$, and the second is the same computation with c and s interchanged. Dividing by $2s$ and by $2c$ respectively,

$$\frac{H(1-c)}{s} = \frac{1-c+s}{2} = \beta, \qquad \frac{H(1-s)}{c} = \frac{1+c-s}{2} = \alpha.$$

The remaining bounds are weaker. Indeed $\beta \leq H$ because $-c \leq c$; and $\beta \leq H(1+s)/c$ because

$$c(1-c+s) \leq c(1+s) \leq (1+s) \leq (1+c+s)(1+s),$$

using $0 < c \leq 1$. Symmetrically $\alpha \leq H$ because $-s \leq s$, and $\alpha \leq H(1+c)/s$ because $s(1+c-s) \leq s(1+c) \leq (1+c) \leq (1+c+s)(1+c)$. Hence

$$W \cap \{x_1 = H\} = \{H\} \times [-\alpha, \beta],$$

a nonempty segment since $-\alpha \le \beta$ (equivalently $-1 \le 1$).

Finally both endpoints lie in Z: in the coordinates fixed above,

$$\tfrac12(e+e'+f-f') = \tfrac12\bigl(1+c+s,\ 0+s+1-c\bigr) = (H,\beta),$$

$$\tfrac12(e+e'-f-f') = \tfrac12\bigl(1+c+s,\ 0+s-1-c\bigr) = (H,-\alpha),$$

and each is a point $\sum_j \lambda_j g_j$ with every $\lambda_j = \pm\frac12$, hence lies in $Z = \mathrm{Zon}(e,f,e',f')$ (Proposition 4.38). Since Z is convex, the whole segment joining them lies in Z:

$$W \cap \{x_1 = H\} \subseteq Z.$$

Step 5: the radial argument. Let $x \in W$ with $x \ne 0$; we show $x \in Z$. Since $\langle x, e\rangle = x_1$ and $\langle x, -e\rangle = -x_1$, and similarly for f,

$$m := \max_{n\in N}\langle x,n\rangle \ \ge\ \max\bigl(|x_1|,|x_2|\bigr) > 0,$$

a maximum over a finite set; and $m \le H$ because $x \in W$. Fix $n^* \in N$ with $\langle x, n^*\rangle = m$ and set

$$y := \frac{H}{m}\,x.$$

Then $\langle y,n\rangle = \frac{H}{m}\langle x,n\rangle \le \frac{H}{m}\,m = H$ for every $n \in N$, since $H/m > 0$; so $y \in W$. And $\langle y, n^*\rangle = H$, so y lies on the facet line of n^*.

We claim $y \in Z$. If $n^* = e$ this is Step 4. The other seven cases reduce to this one by two symmetries.

(a) Quarter turns. Fix either sign σ, and put $\varepsilon := \sigma\det(e,f) \in \{-1,+1\}$. Then

$$J_\sigma e = \varepsilon f, \qquad J_\sigma f = -\varepsilon e:$$

indeed $J_\sigma e$ is a unit vector orthogonal to e, hence $\pm f$, and the sign is decided by $\langle J_\sigma e, f\rangle = \sigma\det(e,f) = \varepsilon$; then $f = \varepsilon J_\sigma e$, so $J_\sigma f = \varepsilon J_\sigma^2 e = -\varepsilon e$. So J_σ permutes $\{e,f,-e,-f\}$ in the 4-cycle

$$e \ \mapsto\ \varepsilon f \ \mapsto\ -e \ \mapsto\ -\varepsilon f \ \mapsto\ e,$$

and likewise permutes $\{e',f',-e',-f'\}$ in a 4-cycle; so $J_\sigma N = N$. Since J_σ maps the axes of F to axes of F, Proposition 4.18(iii) gives $J_\sigma Q_F = Q_F$, and likewise $J_\sigma Q_G = Q_G$; hence $J_\sigma Z = Z$ (a linear map takes a Minkowski sum to the sum of the images). Also $J_\sigma W = W$: for $x \in W$ and $n \in N$,

$$\langle J_\sigma x, n\rangle = \langle x, J_\sigma^{\mathsf T} n\rangle = \langle x, -J_\sigma n\rangle \le H,$$

using $J_\sigma^{\mathsf T} = -J_\sigma$ and $-J_\sigma n \in N$; and J_σ is invertible with $J_\sigma^{-1} = -J_\sigma$, whose action on N is likewise inside N. Finally $\langle J_\sigma u, J_\sigma v\rangle = \langle u,v\rangle$, so $\langle J_\sigma x, J_\sigma n^*\rangle = \langle x,n^*\rangle = m$: applying J_σ moves the maximising normal along the 4-cycle. Hence, given $y \in W$ whose maximising normal lies in $\{\pm e, \pm f\}$, there is $k \in \{0,1,2,3\}$ with J_σ^k carrying that normal to e; then $J_\sigma^k y \in W$ has maximising normal e, so $J_\sigma^k y \in Z$ by Step 4, and therefore $y = J_\sigma^{-k}(J_\sigma^k y) \in Z$.

(b) Exchanging the two frames. The data $Z = Q_F + Q_G = Q_G + Q_F$, N, H and W are symmetric in F and G. Moreover the normalisation of Step 0 holds with the roles exchanged and with the same c, s: putting $\tilde e = e'$, $\tilde f = -f'$ (a frame whose axes are those of G) one computes

$$c\,\tilde e + s\,\tilde f = c(c,s) + s(s,-c) = (c^2+s^2,\ cs - sc) = (1,0) = e,$$

$$-s\,\tilde e + c\,\tilde f = -s(c,s) + c(s,-c) = (0,-1) = -f,$$

so with $\tilde e' = e$ and $\tilde f' = -f$, a frame whose axes are those of F, the pair $\bigl((\tilde e,\tilde f),(\tilde e',\tilde f')\bigr)$ satisfies the hypotheses of Step 0, with the same $c \ge 0$ and $s \ge 0$. The case $n^* = \tilde e = e'$ of the exchanged configuration is Step 4 for that configuration, and combining with (a) applied there, every $n^* \in \{\pm e', \pm f'\}$ is covered.

So $y \in Z$ in all cases. Since $0 \in Z$ and Z is convex, and $x = (m/H)y$ with $m/H \in (0,1]$, the point x lies on the segment $[0,y] \subseteq Z$. Together with $0 \in Z$ this gives $W \subseteq Z$, and with Step 2, $Z = W$.

The interior. W is a finite intersection of half-planes with unit normals, so Lemma 4.13 gives $\operatorname{int} W = \{x : \langle x,n\rangle < H\ \forall n \in N\}$, which is the second display. □

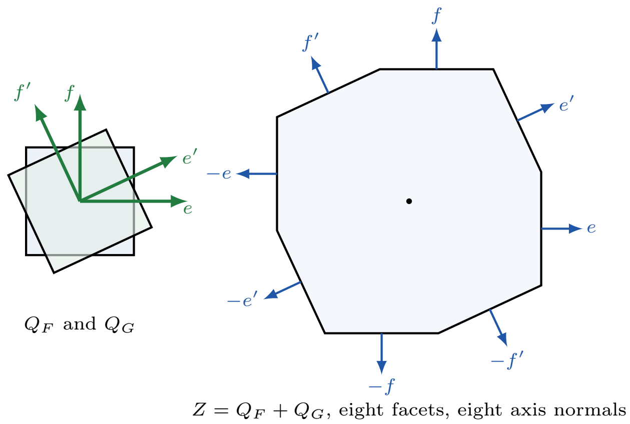


Figure 4.4: The Minkowski sum of two unit squares with $t = 25°$. Each edge of Z is a translate of one of the four generating segments, in the cyclic order $e, e', f, f', -e, -e', -f, -f'$, and each outward normal (blue) is the quarter turn of the edge it borders, hence an axis of one of the two green frames. The four green axes are drawn at their true length 1, so they protrude beyond the squares of side 1. All eight facets sit at the same distance $H = (1 + \cos t + \sin t)/2$ from the centre.

Theorem 4.40 (an owned separating normal). *Let $Q(F, P)$ and $Q(G, Q)$ be unit squares with disjoint interiors, and let t and $H = (1 + \cos t + \sin t)/2$ be as in Theorem 4.39. Then there is a unit vector n, which is an axis of F or an axis of G, with*

$$\langle Q - P,\ n\rangle \ \geq\ w_F(n) + w_G(n) \ =\ H \ \geq\ 1.$$

Proof. By Theorem 4.29, $Q - P \notin \operatorname{int} Z$. By the second display of Theorem 4.39, some $n \in N$ fails the strict inequality, that is, satisfies $\langle Q - P, n\rangle \geq w_F(n) + w_G(n)$; and every element of N is an axis of F or of G. The value of $w_F(n) + w_G(n)$ is H for every $n \in N$, and $H \geq 1$, both by Step 1 of Theorem 4.39. □

Each of the eight candidate normals is an axis of one of the two frames, so once the frames are frozen (Chapter 9) the normal separating a given pair is supplied by one of the two squares, its *owner*. By Step 1 of Theorem 4.39 all eight thresholds are the same number $H = (1 + c + s)/2$, so knowing which square supplies the normal carries no metric information; what it carries is combinatorial, namely that two branches owned by one square use axes of a single frame, and hence are parallel or perpendicular. Chapters 10 to 12 use this.

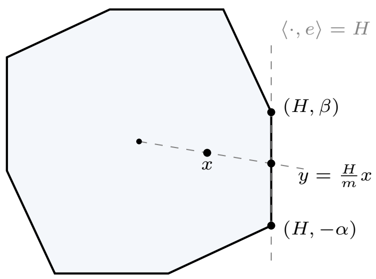


Figure 4.5: Step 5 of Theorem 4.39. A point $x \in W$ is scaled radially until it reaches the facet line of its maximising normal (here e); Step 4 identifies the slice of W on that line as the segment between the two marked vertices of Z, so the scaled point lies in Z, and hence so does x, by convexity and $0 \in Z$.

Example 4.41 (the regular octagon). Take $t = \pi/4$, so $c = s = 1/\sqrt{2}$ and $H = (1 + \sqrt{2})/2$. The four generators e, e', f, f' are unit vectors at angles $0, \pi/4, \pi/2, 3\pi/4$, so the boundary of Z consists of eight unit segments whose directions turn by $\pi/4$ at each step: Z is a regular octagon of side

1. Its eight facets are at distance $H = (1+\sqrt{2})/2 \approx 1.207$ from the centre, in agreement with Step 1. At the other extreme $t = 0$ we get $Z = 2Q_F$, the square of side 2, whose four facets are at distance $H = 1$. In between, Z is an octagon with edges alternating in the pattern e, e', f, f' and facet distance $H = (1 + \cos t + \sin t)/2$ increasing from 1 to $(1 + \sqrt{2})/2$ as t runs from 0 to $\pi/4$.

Remark 4.42 (general zonogons). For pairwise non-parallel nonzero $g_1, \ldots, g_k$ one can show that $\mathrm{Zon}(g_1, \ldots, g_k)$ is a centrally symmetric convex polygon with exactly $2k$ edges, whose edge directions are $\pm g_1, \ldots, \pm g_k$ and whose outward facet normals are the quarter turns $\pm J_\sigma g_j/\|g_j\|$; the boundary is traced by laying the segments head to tail in order of increasing direction angle. Only the case $k = 4$ with the generators the axes of two frames is proved here; the general theory, and its higher-dimensional form for zonotopes, are not developed. Exercise 4.7 computes the area of a general zonogon.

Remark 4.43 (what later chapters may quote). Later chapters use only the following results from this chapter.

(E1) Lemma 4.13 (interior of a finite intersection of half-planes with unit normals) and Lemma 4.12.

(E2) Proposition 4.15 (interior implies strict in every direction) and Proposition 4.31 ($h_{Q_F} = w_F$, attained at an explicit vertex).

(E3) Proposition 4.32(i) (additivity of support functions) and Corollary 4.33 ($h_Z = w_F + w_G$).

(E4) Theorem 4.29 (the obstacle theorem) and Theorem 4.39 (the octagon theorem, including the equal threshold H).

(E5) Theorem 4.40, which is Chapter 5's owned-branch lemma.

Exercises

Exercise 4.1. (a) Give two convex sets whose union is not convex.

(b) Show that if X is convex then so is $\operatorname{int} X$. (Suggestion: if $B(x, \rho) \subseteq X$ and $B(y, \rho') \subseteq X$, consider $B\big((1-\lambda)x + \lambda y,\ (1-\lambda)\rho + \lambda\rho'\big)$.)

(c) Show that $\operatorname{conv} X$ is convex directly from Definition 4.8, without using Lemma 4.3.

Exercise 4.2. Let $F = (e, f)$ be a frame and $P \in \mathbb{R}^2$.

(a) For each of the four axes $n \in \{\pm e, \pm f\}$, compute the facet $\{X \in Q(F, P) : \langle X - P, n\rangle = \frac{1}{2}\}$ of the presentation of $Q(F, P)$ by four half-planes, and show that it is a segment of length 1 joining two of the four vertices. (In particular all four facets are edges, and every edge normal is an axis.)

(b) Deduce from Lemma 4.19 that a unit square determines its frame exactly up to quarter turns and sign changes, and explain why this makes the set of frames a circle of circumference $\pi/2$ rather than 2π.

Exercise 4.3. (a) With F the standard frame and $n = (\cos\theta, \sin\theta)$, compute $w_F(n)$, find its maximum and minimum over $\theta \in [0, 2\pi)$, and say which θ attain them.

(b) Prove $w_F(n) \geq \frac{1}{2}$ for every frame F and every unit vector n, directly from $\langle n, e\rangle^2 + \langle n, f\rangle^2 = 1$, and determine the equality case.

(c) Prove that the set $\{\langle X, n\rangle : X \in Q(F, 0)\}$ equals $[-w_F(n), w_F(n)]$, and interpret $2w_F(n)$ as the length of a shadow. (For the "every intermediate value is attained" part, convexity and the intermediate value theorem in one variable may be used.)

Exercise 4.4. (a) Compute h_X for $X = [-g, g]$ (a segment through 0), for the closed ball $X = \bar{B}(0, \rho)$ and for the open ball $X = B(0, \rho)$, and for $X = \operatorname{conv}\{V_1, V_2, V_3\}$. In which of the four cases is the supremum attained?

(b) Let $X = Q_F$ and $Y = \bar{B}(0, \rho)$. Verify $h_{X+Y} = h_X + h_Y$ by computing both sides, and describe $X + Y$ geometrically. Then check that the same identity holds with $Y = B(0, \rho)$, where neither side is attained; the proof of Proposition 4.32(i) did not use attainment.

(c) Prove that h_X is positively homogeneous, $h_X(\lambda n) = \lambda h_X(n)$ for $\lambda \geq 0$, and subadditive, $h_X(m + n) \leq h_X(m) + h_X(n)$. Give an example of a bounded X for which $h_X(-n) \neq h_X(n)$ for some n.

Exercise 4.5 (two obstacles)**.** Throughout, F is the standard frame $e = (1, 0)$, $f = (0, 1)$. First let $G = F$.

(a) Show directly, without quoting Theorem 4.39, that $Z = Q_F + Q_G$ is the closed square of side 2 centred at 0.
(b) Deduce from Theorem 4.29 that two axis-parallel unit squares with centres P, Q have disjoint interiors if and only if the two centres differ by at least 1 in the first coordinate or by at least 1 in the second. Use this to re-prove $\|Q - P\| \geq 1$ in this case, and check that the three centres $(0,0), (1,0), (0,1)$ are pairwise admissible.

Now let G have axes $\pm e', \pm f'$ with $e' = (\cos\frac{\pi}{6}, \sin\frac{\pi}{6})$ and $f' = (-\sin\frac{\pi}{6}, \cos\frac{\pi}{6})$, so that $t = \pi/6$ in Theorem 4.39.

(c) Compute H and verify $H \geq 1$. Then run Step 4 of the proof of Theorem 4.39 with these numbers: compute α and β, write down the two endpoints of the facet $W \cap \{x_1 = H\}$, and check that they are the two points $\frac{1}{2}(e + e' - f') \pm \frac{1}{2}f$, each of which lies in Z because all four coefficients are $\pm\frac{1}{2}$.
(d) Show that $\alpha + \beta = 1$ for every $t \in (0, \pi/2)$, so that the facet with normal e always has length 1, and identify it as a translate of the generating segment $[-\frac{1}{2}f, \frac{1}{2}f]$.
(e) Two unit squares $Q(F, P)$ and $Q(G, Q)$ are given. Decide, by testing the eight inequalities of Theorem 4.39, whether their interiors meet, in each of the two cases $Q - P = (\frac{6}{5}, -\frac{1}{5})$ and $Q - P = (1, \frac{1}{2})$. In the separated case, say which of the eight axes supplies the separating inequality and which of the two squares owns it. (Exact arithmetic is quicker than decimals: for instance $\frac{6}{5} \geq H$ reduces to $81 \geq 75$.)

Exercise 4.6 (a square and a disc)**.** Let F be a frame, let $\rho > 0$ and let $D = \bar{B}(0, \rho)$, so that $\operatorname{int} D = B(0, \rho)$.

(a) Check that D is convex, bounded, centrally symmetric and has 0 in its interior.
(b) Prove that for all P, Q,

$$\operatorname{int} Q(F, P) \cap \operatorname{int}\big(Q + D\big) \neq \emptyset \iff Q - P \in \operatorname{int}(Q_F + D),$$

by following the proof of Theorem 4.29 line by line and recording which properties of Q_G are used. (Only the properties in part (a) are needed.)
(c) Identify $Q_F + D$ for $\rho = \frac{1}{2}$ as the rounded square of Example 4.27, and compute its support function. Say why Theorem 4.39 has no analogue here: which feature of the four generating segments made the obstacle a polygon, and what replaces it when one body is a disc?

Exercise 4.7 (zonogon area)**.** This exercise assumes two facts about area that are not proved in this book: congruent polygons have equal area, and area is additive over dissections of a polygon into finitely many pieces with pairwise disjoint interiors.

Let $g_1, \dots, g_k \in \mathbb{R}^2$ be nonzero and pairwise non-parallel.

(a) Do the case $k = 2$ first: show that $\operatorname{Zon}(g_1, g_2)$ is a parallelogram of area $|\det(g_1, g_2)|$.
(b) Prove that the area of $\operatorname{Zon}(g_1, \dots, g_k)$ is $\sum_{1 \leq i < j \leq k} |\det(g_i, g_j)|$. (Induct on k: adding the segment $[-\frac{1}{2}g_k, \frac{1}{2}g_k]$ to a convex polygon Y adds a family of parallelograms along the boundary of Y, one per edge of Y, of total area $\sum_{j<k} |\det(g_j, g_k)|$.)
(c) Deduce that for $Z = Q_F + Q_G$ with $\langle e', e\rangle = \cos t$, $\langle e', f\rangle = \sin t$ and $t \in (0, \pi/2)$, the area of Z is $2 + 2(\cos t + \sin t)$. (The interval is open because at $t = 0$ and $t = \pi/2$ the four generators e, f, e', f' merge into two parallel pairs, and the hypothesis of (b) fails.)
(d) Check separately, by direct computation, that the same formula gives the correct value at the two excluded endpoints $t = 0$ and $t = \pi/2$, where Z is the square of side 2; and check the formula at $t = \pi/4$ against the regular octagon of side 1, whose area is $2(1 + \sqrt{2})$.

Exercise 4.8 (four points, for Chapter 13)**.** Let $p_0, p_1, p_2, p_3 \in \mathbb{R}^2$ be such that the four *turn determinants*

$$\det\big(p_{j+1} - p_j,\ p_{j+2} - p_{j+1}\big) \qquad (j = 0, 1, 2, 3,\ \text{indices mod } 4)$$

are all strictly positive.

(a) Prove that $\operatorname{conv}\{p_0, p_1, p_2, p_3\}$ is the intersection of the four half-planes $\{x : \det(p_{j+1} - p_j,\ x - p_j) \geq 0\}$.
(b) Prove that each p_j is a vertex of this polygon in the sense of Definition 4.6, and in particular that no p_j lies in the convex hull of the other three.
(c) Prove that the area of the quadrilateral equals $\frac{1}{2}|\det(p_2 - p_0,\ p_3 - p_1)|$, half the absolute determinant of the two diagonals.

(d) Exhibit four points for which the four turn determinants are nonzero but not all of the same sign, and show that the conclusion of (a) fails for them.

Chapter 5

Separating Branches, Ownership, and the Threshold

This chapter assumes.

- From Chapter 1: the unit square $Q(F, P)$ with frame $F = (e, f)$ and centre P; the description of $\operatorname{int} Q(F, P)$ by the two strict slab inequalities; the ball $B(x, \rho)$ and the interior $\operatorname{int} X$.
- From Chapter 2: the determinant $\det(u, v) = u_1v_2 - u_2v_1$; the quarter turn J_σ with $\langle J_\sigma u, v\rangle = \sigma \det(u, v)$ and $\langle J_\sigma u, J_\sigma v\rangle = \langle u, v\rangle$; the directed-angle lemma (Lemma 2.3), which for unit vectors u, v produces a unique $\gamma \in [0, 2\pi)$ with $\langle u, v\rangle = \cos\gamma$ and $\sigma \det(u, v) = \sin\gamma$; and the unit-vector companion $|a| + |b| \geq 1$ whenever $a^2 + b^2 = 1$.
- From Chapter 4, exactly five items, and no others:
  - (C1) Lemma 4.13: the interior of a finite intersection of half-planes with unit normals is cut out by the corresponding strict inequalities. (Together with Lemma 4.12: a nonzero linear functional never attains its maximum over a set at an interior point.)
  - (C2) Proposition 4.15: writing $w_F(n) = \big(|\langle n, e\rangle| + |\langle n, f\rangle|\big)/2$, one has $\langle X - P, n\rangle \leq w_F(n)$ for every $X \in Q(F, P)$ and every unit vector n, with strict inequality when $X \in \operatorname{int} Q(F, P)$. Proposition 4.31 adds that the bound is attained, at an explicit vertex.
  - (C3) Proposition 4.32(i): support functions are additive under Minkowski sums, $h_{X+Y} = h_X + h_Y$; and Corollary 4.33: $h_Z(n) = w_F(n) + w_G(n)$ for $Z = Q(F, 0) + Q(G, 0)$.
  - (C4) Theorem 4.39 (the octagon theorem): with N the set of axes of F and of G (four or eight unit vectors),
  $$Z = \big\{x : \ \langle x, n\rangle \leq w_F(n) + w_G(n) \ \text{ for all } n \in N\big\},$$
  $$\operatorname{int} Z = \big\{x : \ \langle x, n\rangle < w_F(n) + w_G(n) \ \text{ for all } n \in N\big\},$$
  and all of the numbers $w_F(n)+w_G(n)$, $n \in N$, are equal to one number $(1+\cos\tau+\sin\tau)/2 \geq 1$, where the axes have been relabelled so that $e' = e\cos\tau + f \sin\tau$ with $\tau \in [0, \pi/2]$.
  - (C5) Theorem 4.29 (the obstacle theorem): $\operatorname{int} Q(F, P) \cap \operatorname{int} Q(G, P') \neq \emptyset$ if and only if $P' - P \in \operatorname{int} Z$.

  No general polygon-facet duality is used. The reverse inclusion "a convex set is the intersection of its supporting half-planes" is a separation theorem, not available before Chapter 8 (Remark 4.35); for the body considered here, (C4) supplies eight explicit normals.
- Nothing from Chapter 3 is used.

Notation. The two centres of this chapter are P and P'; the letter Q is reserved for the square constructor $Q(F, P)$. (Chapter 4 wrote the second centre Q; nothing but the letter changes.) In figures the displacement $P' - P$ is drawn in black and the branch normals in blue.

The orientation sign $\sigma \in \{-1, +1\}$ occurs in this chapter only inside proofs; every statement below is independent of which sign is chosen.

5.1 The projected half-width and the shadow of a square

To compare two squares along a direction n we need only know how long each square's shadow on the line $\mathbb{R}n$ is. The two endpoints of that shadow were computed in Chapter 4; it remains to name the resulting quantity and to check that the shadow is the interval between them.

Definition 5.1 (Projected half-width)**.** Let $F = (e, f)$ be a frame and n a unit vector. The *projected half-width* of F in direction n is

$$w_F(n) \;:=\; \frac{|\langle n, e\rangle| + |\langle n, f\rangle|}{2}.$$

Lemma 5.2 (Shadow lemma)**.** *Let $F = (e, f)$ be a frame, P a point, and n a unit vector. Then*

$$\{\, \langle X, n\rangle \;:\; X \in Q(F, P) \,\} \;=\; \big[\, \langle P, n\rangle - w_F(n),\; \langle P, n\rangle + w_F(n) \,\big],$$

an interval of length $2w_F(n)$ centred at $\langle P, n\rangle$. In particular $h_{Q(F,0)}(n) = w_F(n)$. Moreover, if $X \in \operatorname{int} Q(F, P)$ then $\langle X - P, n\rangle < w_F(n)$ strictly.

Proof. The two bounds $|\langle X - P, n\rangle| \le w_F(n)$ and the strict version are (C2), applied to n and to $-n$ and using $w_F(-n) = w_F(n)$. The upper bound is attained at the vertex $P + \frac{1}{2}\big(\operatorname{sgn}\langle n, e\rangle\, e + \operatorname{sgn}\langle n, f\rangle\, f\big)$ of Proposition 4.31, and the lower bound at the antipodal vertex; taking $P = 0$ gives $h_{Q(F,0)}(n) = w_F(n)$.

Only the surjectivity onto the whole interval is new. The square is convex, so the segment joining a minimiser X_0 to a maximiser X_1 lies in $Q(F, P)$; the map $t \mapsto \langle X_0 + t(X_1 - X_0), n\rangle$ is continuous on $[0, 1]$, so by the intermediate value theorem it takes every value between the two endpoints. $\square$

Example 5.3 (The axis-parallel square)**.** Let $F = (e_1, e_2)$ be the axis-parallel frame and $n = (\cos\theta, \sin\theta)$, so that $w_F(n) = (|\cos\theta| + |\sin\theta|)/2$ as in Example 4.20. Numerically:

θ	0	$\pi/6$	$\pi/4$
$w_F(n)$	$\frac{1}{2}$	$\frac{\sqrt{3}+1}{4} \approx 0.6830$	$\frac{\sqrt{2}}{2} \approx 0.7071$
$2w_F(n)$	1	$\frac{\sqrt{3}+1}{2} \approx 1.3660$	$\sqrt{2} \approx 1.4142$

Both extremes match the picture: the shadow of a unit square on a line parallel to a side has length 1, and its shadow on a diagonal direction has length $\sqrt{2}$, the length of the diagonal.

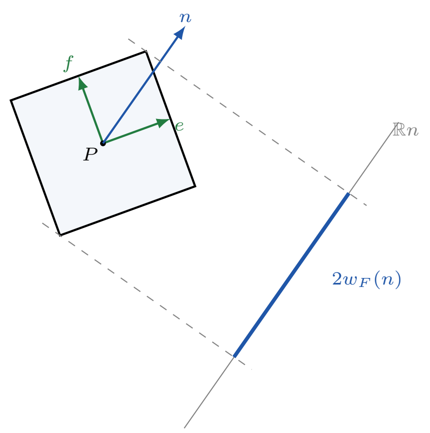


Figure 5.1: The shadow of a unit square. The square has frame (e, f) (green) and centre P; the two support lines perpendicular to the unit normal n (blue) pass through opposite vertices, and cut out on the line $\mathbb{R}n$ an interval of length $2w_F(n)$ centred at the projection of P.

5.2 Four facts about the projected half-width

The following properties of w are used repeatedly below and in later chapters.

Lemma 5.4 (Properties of the projected half-width)**.** *Let $F = (e, f)$ and G be frames and let n be a unit vector.*

(i) (Evenness and axis-dependence.) $w_F(-n) = w_F(n)$. *Moreover w_F depends only on the axis set $\{\pm e, \pm f\}$: if F' is any frame with the same axis set then $w_{F'} = w_F$ as functions on unit vectors.*

(ii) (Value on an axis.) *If n is an axis of F then $w_F(n) = \frac{1}{2}$.*

(iii) (Other-frame formula.) *If m is any axis of G then*

$$w_G(n) = \frac{|\langle n, m\rangle| + |\det(n, m)|}{2},$$

and the right-hand side is the same for all four choices of the axis m.

(iv) (Universal bound.) $w_G(n) \geq \frac{1}{2}$ *for every unit vector n, with equality if and only if n is an axis of G.*

Proof. (i) The defining expression involves n only through $|\langle n, e\rangle|$ and $|\langle n, f\rangle|$, which are unchanged when n is replaced by $-n$. A frame with axis set $\{\pm e, \pm f\}$ is an ordered pair (a, b) of perpendicular axes, so $\{a, b\} = \{\pm e, \pm f\}$ up to signs and order; the expression $(|\langle n, a\rangle| + |\langle n, b\rangle|)/2$ is symmetric in its two terms and even in each of a and b, hence unchanged.

(ii) If $n = \pm e$ then $|\langle n, e\rangle| = 1$ and $|\langle n, f\rangle| = 0$, so $w_F(n) = \frac{1}{2}$; symmetrically for $n = \pm f$.

(iii) Fix either sign $\sigma \in \{-1, +1\}$ and set $m' = J_\sigma m$. Since J_σ is a rotation it preserves norms, so $\|m'\| = 1$, and $\langle m', m\rangle = \langle J_\sigma m, m\rangle = \sigma \det(m, m) = 0$; thus (m, m') is a frame. Its axes are $\pm m, \pm m'$, and these are the axes of G: writing $G = (g_1, g_2)$, the axis m is one of $\pm g_1, \pm g_2$, and m' is a unit vector perpendicular to m, hence equal to $\pm$ the other axis. By part (i) we may compute w_G from the frame (m, m'):

$$w_G(n) = \frac{|\langle n, m\rangle| + |\langle n, m'\rangle|}{2}.$$

Now $\langle n, m'\rangle = \langle J_\sigma m, n\rangle = \sigma \det(m, n) = -\sigma \det(n, m)$, so $|\langle n, m'\rangle| = |\det(n, m)|$, which is the stated formula. Independence of the choice of axis: replacing m by $-m$ changes nothing, since both terms are absolute values; replacing m by m' swaps the two terms, because $|\langle n, m'\rangle| = |\det(n, m)|$ as just shown, while

$$\sigma \det(n, m') = \langle J_\sigma n, J_\sigma m\rangle = \langle n, m\rangle$$

(using orthogonality of J_σ), so $|\det(n, m')| = |\langle n, m\rangle|$. The four axes of G are $\pm m, \pm m'$, so all cases are covered.

(iv) With m, m' as above, put $a = \langle n, m\rangle$ and $b = \langle n, m'\rangle$; these are the coordinates of n in the orthonormal basis (m, m'), so $a^2 + b^2 = 1$ and, by the companion inequality of Chapter 2, $|a| + |b| \geq 1$ with equality if and only if $ab = 0$. Hence $w_G(n) \geq \frac{1}{2}$, with equality exactly when $n \perp m'$ or $n \perp m$, i.e. exactly when $n \in \{\pm m, \pm m'\}$. (For G the standard frame this is Example 4.20; part (i) removes the restriction to that frame.) □

Corollary 5.5 (Universal threshold bound)**.** *For all frames F, G and every unit vector n,*

$$w_F(n) + w_G(n) \;\geq\; 1,$$

with equality if and only if n is simultaneously an axis of F and an axis of G.

Proof. Add the two instances of Lemma 5.4(iv). Equality forces $w_F(n) = w_G(n) = \frac{1}{2}$, which by the equality clause of (iv) says that n is an axis of each frame; conversely, if it is, part (ii) gives $\frac{1}{2} + \frac{1}{2}$. □

The bound holds for every unit vector, whether or not it is owned; ownership is what pins down the exact value.

5.3 Oriented separating branches

Definition 5.6 (Branch and threshold)**.** Given frames F, G and a unit vector n, the *threshold* in direction n is

$$H(F, G; n) \;:=\; w_F(n) + w_G(n).$$

An *oriented separating branch* with normal n for the ordered pair of squares $Q(F, P)$, $Q(G, P')$ is the inequality

$$\langle P' - P,\; n\rangle \;\geq\; H(F, G; n). \tag{5.1}$$

Three observations, the third recorded as a lemma.

(1) *A branch is one linear inequality in the centres.* With F, G and n held fixed, $H(F, G; n)$ is a constant, and the left-hand side of (5.1) is a linear function of the pair $(P, P') \in \mathbb{R}^2 \times \mathbb{R}^2$. A condition relating two planar bodies is thereby expressed as a half-space condition on a single point of $\mathbb{R}^4$. In Chapter 6 the normalisation $P_1 = 0$ makes the resulting functions linear rather than affine, which matters there.

(2) *Symmetries of the threshold.* $H(F, G; n) = H(G, F; n)$ from the definition, and $H(F, G; -n) = H(F, G; n)$ by Lemma 5.4(i).

(3) *A branch may be read from either end*, on reversing the normal. This is Lemma 5.7.

Lemma 5.7 (Reorientation)**.** *Let F, G be frames, let P, P' be points, and let n be a unit vector. Then (5.1) holds for the ordered pair $\big(Q(F, P),\, Q(G, P')\big)$ with normal n if and only if it holds for the reversed pair $\big(Q(G, P'),\, Q(F, P)\big)$ with normal $-n$.*

Proof. The branch for the reversed pair with normal $-n$ reads $\langle P - P', -n\rangle \geq H(G, F; -n)$. The left-hand side equals $\langle P' - P, n\rangle$ and, by the two symmetries above, the right-hand side equals $H(F, G; n)$. □

Chapter 9 uses this lemma to orient the three frozen branches cyclically, so that branch i reads $\langle d_i, n_i\rangle \geq H_i$ with $d_i = P_{i+1} - P_i$.

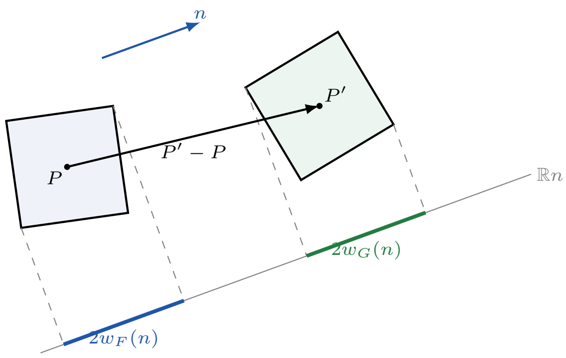


Figure 5.2: An oriented separating branch. The two squares project onto the line $\mathbb{R}n$ to intervals of lengths $2w_F(n)$ and $2w_G(n)$ (support lines dashed), and (5.1) says that the first interval ends before the second begins. The normal n is an axis of a frame, and in general is not parallel to $P' - P$.

5.4 Why a branch suffices, and what it throws away

Proposition 5.8 (A branch is shadow separation)**.** *Let I_P and $I_{P'}$ be the shadows of $Q(F, P)$ and $Q(G, P')$ on the line $\mathbb{R}n$ furnished by Lemma 5.2, that is,*

$$I_P = \big[\langle P, n\rangle - w_F(n),\; \langle P, n\rangle + w_F(n)\big],$$
$$I_{P'} = \big[\langle P', n\rangle - w_G(n),\; \langle P', n\rangle + w_G(n)\big].$$

Then (5.1) holds if and only if $\sup I_P \leq \inf I_{P'}$.

Proof. $\sup I_P \le \inf I_{P'}$ says $\langle P, n\rangle + w_F(n) \le \langle P', n\rangle - w_G(n)$, which rearranges to $\langle P' - P, n\rangle \ge w_F(n) + w_G(n)$. □

Two closed intervals of $\mathbb{R}$ will be said to have *disjoint interiors* when $\sup I \le \inf I'$ or $\sup I' \le \inf I$; the proposition says that a branch with normal n asserts that the two shadows have disjoint interiors, in the order given by n. (For intervals this is a definition; it is not the interior operator of $\mathbb{R}^2$.)

Theorem 5.9 (Sufficiency of a branch)**.** *If* (5.1) *holds for some unit vector n, then*

$$\operatorname{int} Q(F, P) \cap \operatorname{int} Q(G, P') = \emptyset.$$

Proof. This is Corollary 4.16(ii): a common interior point X would give $\langle X - P, n\rangle < w_F(n)$ and $\langle P' - X, n\rangle < w_G(n)$ by (C2), and adding the two strict inequalities contradicts (5.1). □

Remark 5.10 (A branch depends only on the two shadows). By Proposition 5.8, the branch condition depends on the four data (F, P, G, P') only through the two shadow intervals on $\mathbb{R}n$; it depends on nothing else about the squares. Two consequences follow.

- For a fixed n the converse of Theorem 5.9 fails: two squares can have disjoint interiors while their shadows in direction n overlap. See Example 5.11.
- Quantifying over n restores the converse. Combining Theorem 5.9 with Lemma 5.14 below, two unit squares have disjoint interiors if and only if some branch holds.

Example 5.11 (The converse fails direction by direction)**.** Let $F = G$ be the axis-parallel frame and take centres $P = (0, 0)$, $P' = (1, \frac{1}{5})$. The shadows in direction e_1 are $[-\frac{1}{2}, \frac{1}{2}]$ and $[\frac{1}{2}, \frac{3}{2}]$, so by Theorem 5.9 the interiors are disjoint. Now test three directions, using $w_F(n) = w_G(n) = (|\cos\theta| + |\sin\theta|)/2$ from Example 5.3:

n	$H(F, G; n)$	$\langle P' - P, n\rangle$	branch
e_1	1	1	holds, with equality
e_2	1	1/5	fails
$(1, 1)/\sqrt{2}$	$\sqrt{2} \approx 1.4142$	$\frac{6}{5\sqrt{2}} \approx 0.8485$	fails

A valid branch exists, but most directions fail to certify the disjointness.

Remark 5.12 (Relaxation to a frozen linear system). Chapter 9 uses the following logical shape. (a) By Theorem 5.9, any configuration of centres satisfying a fixed branch inequality, with frames and normals frozen, has squares with disjoint interiors, so a lower bound proved over the frozen feasible set is a statement about admissible squares. (b) By Lemma 5.14 below, an assumed counterexample lies in such a feasible set. (c) Hence minimising the area of the centre triangle over the frozen linear system is legitimate in both directions: it enlarges the domain, but only within configurations of squares with disjoint interiors.

5.5 The separating axis theorem

The following classical statement explains why only finitely many directions need testing.

Theorem 5.13 (Separating axis theorem; quoted, not proved)**.** *Let X and Y be convex polygons in the plane, with N_X and N_Y their finite sets of outward unit facet normals. Then* $\operatorname{int} X \cap \operatorname{int} Y = \emptyset$ *if and only if there is a unit vector $n \in N_X \cup N_Y$ such that the shadows of X and of Y on the line $\mathbb{R}n$ have disjoint interiors.*

The "if" direction is the argument of Theorem 5.9, and needs only that a point interior to a set satisfies a strict support inequality in every direction: by Lemma 4.12, if $x \in \operatorname{int} X$ then some $y \in X$ has $\langle y, n\rangle > \langle x, n\rangle$, so $\langle x, n\rangle < h_X(n)$. The substantive direction is the other one, whose content is that finitely many candidate directions suffice: one per facet of each polygon, and by evenness only half of these need be tested.

The theorem is not used below. The one instance the main argument requires, two unit squares with the separating constant computed explicitly, is Lemma 5.14, which follows from the octagon theorem (C4). For two axis-parallel unit squares it reduces to the computation of Example 5.15(a).

5.6 The owned separating branch lemma

Throughout this section write

$$Q_F := Q(F,0), \qquad Q_G := Q(G,0),$$
$$Z := Q_F + Q_G = \{\, a+b \;:\; a \in Q_F,\ b \in Q_G \,\},$$

so that Z is the obstacle of Chapter 4: a zonogon, whose interior determines, by (C5), whether the two squares overlap. Both summands are centrally symmetric, $Q_F = -Q_F$ and $\operatorname{int} Q_F = -\operatorname{int} Q_F$ (Lemma 4.24).

Lemma 5.14 (Owned separating branch)**.** *Let $Q(F,P)$ and $Q(G,P')$ be unit squares with disjoint interiors. Then there is a unit vector n which is an axis of F or an axis of G such that*

$$\langle P'-P,\ n\rangle \ \geq\ w_F(n) + w_G(n) = H(F,G;n).$$

Proof. By the obstacle theorem (C5), disjointness of the interiors gives $P'-P \notin \operatorname{int} Z$. By the second display of the octagon theorem (C4), membership of $\operatorname{int} Z$ is the conjunction of the strict inequalities $\langle x,n\rangle < w_F(n)+w_G(n)$ over the finite set N; so some $n \in N$ fails its strict inequality, that is, satisfies $\langle P'-P,n\rangle \geq w_F(n)+w_G(n)$. Every element of N is an axis of F or of G. □

This is Theorem 4.40 in the vocabulary of branches: the separating axis theorem for a pair of unit squares, with the separating constant identified as $w_F(n)+w_G(n)$. Chapters 11 and 12 use that value.

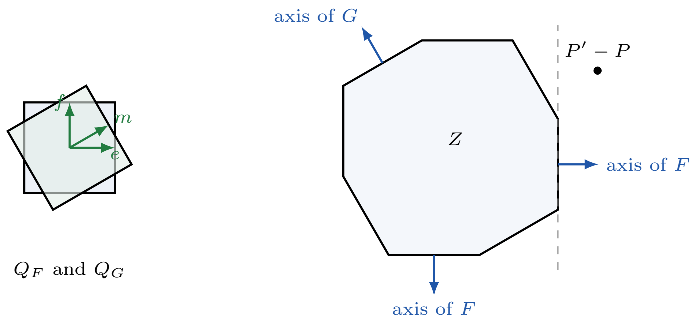


Figure 5.3: Left: the two source squares Q_F (blue) and Q_G (green) at the origin, with the frame (e,f) of F and one axis m of G in green. Right: the obstacle $Z = Q_F + Q_G$. Each edge is parallel to an axis of F or of G, so each of the eight normals of (C4) (blue) is an axis of the same frame. A point $P'-P$ outside Z violates one of the eight inequalities, and that inequality's normal is the normal of an owned branch.

Example 5.15 (The obstacle in two cases)**.** (a) $F = G = (e_1,e_2)$. Then $Q_F = Q_G = [-\frac{1}{2},\frac{1}{2}]^2$ and $Z = [-1,1]^2$, the axis-parallel square of side 2 centred at 0: indeed the coordinates of $a+b$ with $a,b \in [-\frac{1}{2},\frac{1}{2}]^2$ range independently over $[-1,1]$. Its normals are $\pm e_1, \pm e_2$ and $h_Z = 1$ on each of them. So by (C5) and (C1), two axis-parallel unit squares with centres P, P' have disjoint interiors if and only if

$$\max\bigl(|\langle P'-P, e_1\rangle|,\ |\langle P'-P, e_2\rangle|\bigr) \ \geq\ 1,$$

and every owned threshold equals 1. Here the separating axis theorem is unnecessary: the conclusion is immediate.

(b) $F = (e_1,e_2)$ and G the frame at angle $\pi/4$, with axes $\pm m, \pm m'$ where $m = (1,1)/\sqrt{2}$ and $m' = (-1,1)/\sqrt{2}$. Then Z is a sum of four segments in the directions e_1, e_2, m, m', a regular octagon of side 1 (Example 4.41). Its normals are again the four axis directions of F and G, and

$$h_Z(e_1) = w_F(e_1) + w_G(e_1) = \frac{1}{2} + \frac{|\langle e_1, m\rangle| + |\det(e_1,m)|}{2}$$
$$= \frac{1}{2} + \frac{\sqrt{2}}{2} = \frac{1+\sqrt{2}}{2} \approx 1.2071.$$

Since $\frac{1+\sqrt{2}}{2} > 1$, tilted squares must be separated further along an axis than aligned ones; this value is $H(\pi/4)$ of Section 5.9.

5.7 Ownership

Definition 5.16 (Owner)**.** A branch with normal n between $Q(F, P)$ and $Q(G, P')$ is *owned by the first square* if n is, up to sign, an axis of F, and *owned by the second square* if n is, up to sign, an axis of G; it is *owned* if it is owned by at least one of them. When both squares own it, one of the two is designated the owner, once and for all.

In this language, Lemma 5.14 says that a pair of unit squares with disjoint interiors admits an owned branch.

Remark 5.17 (Ownership carries no metric information). By Lemma 5.22 below, an owned branch between squares with frames F and G has threshold $H(d_4(F, G))$ whichever of the two squares supplies the axis. Nothing quantitative distinguishes the two possible owners. In Chapter 12 the branch joining the third square to the first may be owned by the first rather than the third, and its threshold is still $H(d_4)$ of the two frames involved.

Lemma 5.18 (Same owner implies parallel or perpendicular)**.** *If two branches are owned by the same square, with frame $F = (e, f)$, then their unit normals n, n' satisfy $n' = \pm n$ or $\langle n, n'\rangle = 0$. Equivalently, $|\det(n, n')| \in \{0, 1\}$.*

Proof. Both n and n' lie in $\{\pm e, \pm f\}$ and $e \perp f$. If they are both in $\{\pm e\}$ or both in $\{\pm f\}$ then $n' = \pm n$ and $|\det(n, n')| = 0$; otherwise one is $\pm e$ and the other $\pm f$, so $\langle n, n'\rangle = 0$ and, both being unit vectors spanning the plane, $|\det(n, n')| = 1$. □

Chapter 12 uses this: the source of the owner tournament owns two branches, so n_1 and n_3 are axes of a single frame; the parallel case is excluded there by positivity of an oriented sine, leaving $s_3 = 1$.

Given three unit squares with pairwise disjoint interiors, Lemma 5.14 supplies one owned branch per pair; directing each edge of the centre triangle away from the owner of its branch produces a tournament on three vertices. The analysis of that tournament is Chapter 10.

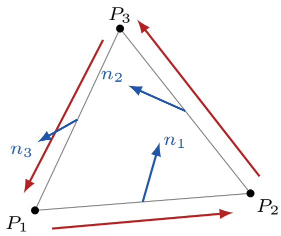


Figure 5.4: The owner tournament on three centres. Red arrows point from the owner of a branch to the other endpoint; they are drawn parallel to the edges, offset. The blue normals n_i are axes of the frozen frames and are in general not perpendicular to the edges d_i.

5.8 The frame circle and the frame distance

Both w and H depend on a frame only through its axis set, which is unchanged by a quarter turn. The natural parameter for a frame is therefore an angle modulo $\pi/2$.

Lemma 5.19 (The frame circle)**.** *Let $F = (e, f)$ be a frame and write $e = (\cos\theta, \sin\theta)$. Then the axis set of F is*

$$\{\, (\cos(\theta + k\pi/2),\ \sin(\theta + k\pi/2)) \ :\ k = 0, 1, 2, 3 \,\}.$$

Consequently the axis set depends only on the class $[\theta] \in \mathbb{T} := \mathbb{R}/(\pi/2)\mathbb{Z}$, and two frames have the same axis set if and only if they determine the same class.

Proof. The vector f is a unit vector perpendicular to e, and the only such vectors are $\pm(-\sin\theta, \cos\theta) = \pm(\cos(\theta+\pi/2), \sin(\theta+\pi/2))$. Also $-e = (\cos(\theta+\pi), \sin(\theta+\pi))$. Hence the four axes $\pm e, \pm f$ are precisely the unit vectors of angles $\theta, \theta+\pi/2, \theta+\pi, \theta+3\pi/2$, which is the displayed set. That set is unchanged if θ is replaced by $\theta+\pi/2$, so it depends only on $[\theta]$.

Conversely, suppose F with $e = (\cos\theta, \sin\theta)$ and F' with $e' = (\cos\theta', \sin\theta')$ have the same axis set. Then e' belongs to the axis set of F, so $(\cos\theta', \sin\theta') = (\cos(\theta+k\pi/2), \sin(\theta+k\pi/2))$ for some k, whence $\theta' \equiv \theta + k\pi/2 \pmod{2\pi}$ and in particular $\theta' \equiv \theta \pmod{\pi/2}$, i.e. $[\theta'] = [\theta]$. ◻

We write $\theta_F \in \mathbb{T}$ for the class of F, and call $\mathbb{T}$ the *frame circle.*

Lemma 5.20 (The frame distance)**.** *For $x, y \in \mathbb{R}$ set*

$$d_4([x],[y]) \;:=\; \min_{k\in\mathbb{Z}} \bigl|x - y - k\pi/2\bigr|.$$

Then:

(a) *the minimum is attained, and the value does not depend on the chosen representatives x, y;*

(b) $0 \le d_4 \le \pi/4$*;*

(c) *d_4 has the three properties of a distance on $\mathbb{T}$: it is symmetric, it vanishes exactly when $[x] = [y]$, and it satisfies the triangle inequality*

$$d_4([x],[z]) \;\le\; d_4([x],[y]) + d_4([y],[z]);$$

(d) *writing $d_4(F,G) := d_4(\theta_F, \theta_G)$, this number equals the least angle between an axis of F and an axis of G; moreover the least angle is attained from every fixed axis of F, that is, for each axis n of F there is an axis m of G with angle exactly $d_4(F,G)$ between n and m.*

Proof. (a) Let $r \in [0, \pi/2)$ be the representative of $x - y$ modulo $\pi/2$, so $x - y = r + k_0\pi/2$ for some $k_0 \in \mathbb{Z}$. Then, as k runs over $\mathbb{Z}$,

$$\begin{aligned}\bigl\{|x-y-k\pi/2| :\; k\in\mathbb{Z}\bigr\} &= \bigl\{|r - j\pi/2| :\; j\in\mathbb{Z}\bigr\}\\ &= \{\, r + j\pi/2 :\; j \ge 0\,\} \;\cup\; \{\,(\pi/2 - r) + j\pi/2 :\; j \ge 0\,\},\end{aligned}$$

whose minimum is $\min(r, \pi/2 - r)$, attained. Changing x or y by a multiple of $\pi/2$ changes $x - y$ by a multiple of $\pi/2$, which merely reindexes the set being minimised.

(b) $r + (\pi/2 - r) = \pi/2$, so $\min(r, \pi/2 - r) \le \pi/4$.

(c) Symmetry: the set $\{|x - y - k\pi/2|\}$ equals $\{|y - x - k\pi/2|\}$ after $k \mapsto -k$. Vanishing: by (a), $d_4 = 0$ iff $\min(r, \pi/2 - r) = 0$ iff $r = 0$ iff $x - y \in (\pi/2)\mathbb{Z}$ iff $[x] = [y]$. Triangle inequality: choose k, ℓ attaining the two minima on the right; then

$$\begin{aligned}d_4([x],[z]) &\le \bigl|x - z - (k+\ell)\pi/2\bigr|\\ &\le \bigl|x - y - k\pi/2\bigr| + \bigl|y - z - \ell\pi/2\bigr| = d_4([x],[y]) + d_4([y],[z]).\end{aligned}$$

(The same argument applies to $\mathbb{R}/H\mathbb{Z}$ for any $H > 0$. The triangle inequality is used in Chapter 12.)

(d) By Lemma 5.19, the axes of F are the unit vectors with angles $\theta_F + k\pi/2$ and those of G the unit vectors with angles $\theta_G + \ell\pi/2$, $k, \ell \in \mathbb{Z}$. The angle in $[0, \pi]$ between the unit vectors of angles α and β is $\arccos\cos(\alpha - \beta)$, i.e. the distance from $\alpha - \beta$ to $2\pi\mathbb{Z}$. Writing $D = \theta_F - \theta_G$, the angle between the axis of angle $\theta_F + k\pi/2$ and the axis of angle $\theta_G + \ell\pi/2$ is therefore the distance from $D + (k-\ell)\pi/2$ to $2\pi\mathbb{Z}$. Now fix k and let ℓ run over $\mathbb{Z}$: then $j := k - \ell$ runs over all of $\mathbb{Z}$, so the minimum over ℓ is

$$\min_{j\in\mathbb{Z}}\; \min_{p\in\mathbb{Z}} \bigl|D + j\pi/2 - 2\pi p\bigr| = \min_{j\in\mathbb{Z}} \bigl|D + j\pi/2\bigr| = d_4(\theta_F, \theta_G),$$

where the first equality holds because $2\pi p = (4p)\pi/2$, so absorbing $-2\pi p$ into j leaves the same set of numbers. This value is independent of k, which is both the claim about the least angle over all pairs and the claim that it is attained from every fixed axis of F. ◻

Lemma 5.21 (Reduction lemma)**.** *For every real* γ*,*

$$|\cos\gamma| + |\sin\gamma| = \cos t + \sin t, \qquad t := d_4([\gamma],[0]) \in [0,\pi/4],$$

where t *is the distance from* γ *to* $(\pi/2)\mathbb{Z}$*.*

Proof. The function $\gamma \mapsto |\cos\gamma|+|\sin\gamma|$ is unchanged by $\gamma \mapsto \gamma+\pi/2$ (which sends $\cos\gamma \mapsto -\sin\gamma$ and $\sin\gamma \mapsto \cos\gamma$) and by $\gamma \mapsto -\gamma$. By Lemma 5.20(a) there is $k \in \mathbb{Z}$ with $\gamma = k\pi/2 \pm t$ and $t \in [0,\pi/4]$; applying the two invariances, the value at γ equals the value at t, which is $\cos t + \sin t$ since cos and sin are nonnegative on $[0,\pi/4]$. ☐

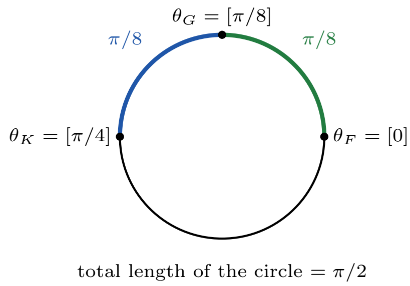


Figure 5.5: The frame circle $\mathbb{T} = \mathbb{R}/(\pi/2)\mathbb{Z}$, drawn with the class $[\theta]$ placed at drawing-angle 4θ so that the whole circle has length $\pi/2$. Three frame classes are marked; the two coloured arcs realise $d_4(F,G) = d_4(G,K) = \pi/8$, and $d_4(F,K) = \pi/4$ is their sum, so the triangle inequality of Lemma 5.20(c) is attained.

5.9 The threshold of an owned branch

Lemma 5.22 (Threshold of an owned branch)**.** *Let* F *and* G *be frames, let* $t = d_4(F,G) \in [0,\pi/4]$*, and let* n *be a unit vector which is an axis of* F *or an axis of* G*. Then*

$$H(F,G;n) \;=\; H(t) \;:=\; \frac{1+\cos t+\sin t}{2},$$

independently of which of the two frames supplies the axis. Moreover $H(t) \ge 1$*, with equality if and only if* $t = 0$*.*

Proof. Since $H(F,G;n) = H(G,F;n)$ and $d_4(F,G) = d_4(G,F)$, we may assume that n is an axis of F. (In different coordinates this is Step 1 of the octagon theorem (C4), where all eight numbers $w_F(n)+w_G(n)$ are shown to coincide.)

By Lemma 5.4(ii), $w_F(n) = \frac{1}{2}$. By Lemma 5.20(d) there is an axis m of G making angle exactly t with n, so $|\langle n,m\rangle| = \cos t$ and $|\det(n,m)| = \sin t$, both nonnegative because $t \in [0,\pi/4]$. Lemma 5.4(iii) then gives $w_G(n) = (\cos t+\sin t)/2$, and adding,

$$H(F,G;n) = \frac{1}{2} + \frac{\cos t+\sin t}{2} = H(t).$$

The bound $H \ge 1$ was noted in Step 1 of (C4): $(\cos t+\sin t)^2 = 1+\sin 2t \ge 1$ and $\cos t+\sin t > 0$ on $[0,\pi/4]$, so $\cos t+\sin t \ge 1$. Equality forces $\sin 2t = 0$ with $2t \in [0,\pi/2]$, i.e. $t = 0$; and $H(0) = 1$. ☐

Corollary 5.23 (Two-normal form)**.** *Let* n *be an axis of* F *and* m *an axis of* G*, and let* $\gamma = \angle_\sigma(n,m)$ *be the directed angle from* n *to* m *in the* σ*-orientation, so that* $\langle n,m\rangle = \cos\gamma$ *and* $\sigma\det(n,m) = \sin\gamma$ *(Lemma 2.3). Then*

$$H(F,G;n) = \frac{1+|\langle n,m\rangle|+|\det(n,m)|}{2} = \frac{1+|\cos\gamma|+|\sin\gamma|}{2} = H(t),$$

where $t = d_4(F,G)$*. Only* $|\det(n,m)|$ *occurs, so the orientation sign is immaterial to the value.*

Proof. The first equality is Lemma 5.4(ii) and (iii) added. The second is the directed-angle lemma together with $|\sigma| = 1$, which gives $|\det(n, m)| = |\sigma \det(n, m)| = |\sin\gamma|$. For the third, $\gamma \equiv \theta_G - \theta_F \pmod{\pi/2}$ by Lemma 5.19, so the distance from γ to $(\pi/2)\mathbb{Z}$ is $d_4(F, G) = t$, and Lemma 5.21 converts $|\cos\gamma| + |\sin\gamma|$ into $\cos t + \sin t$. $\square$

Chapter 11 uses the middle expression, with $n = n_i$, $m = n_{i+1}$ and $\sin\gamma_i > 0$, so that there $H_i = (1 + |\cos\gamma_i| + \sin\gamma_i)/2$.

Proposition 5.24 (The threshold excess). *Define the* excess

$$E(t) := H(t) - 1 = \frac{\cos t + \sin t - 1}{2} \qquad \text{on } [0, \pi/4].$$

Then:

(i) $E(0) = 0$ *and* $E \geq 0$;

(ii) $E'(t) = \dfrac{\cos t - \sin t}{2} \geq 0$ *on* $[0, \pi/4]$, *with equality only at* $t = \pi/4$; *in particular* E *is increasing;*

(iii) $E''(t) = -\dfrac{\cos t + \sin t}{2} < 0$ *on* $[0, \pi/4]$, *so* E *is strictly concave;*

(iv) $E(\pi/4) = \dfrac{\sqrt{2} - 1}{2} \approx 0.2071$, *so* $1 \leq H(t) \leq \dfrac{1 + \sqrt{2}}{2} \approx 1.2071$ *on* $[0, \pi/4]$;

(v) *the function* $\Lambda(t) := (1 + \cos t + \sin t)/2$, *defined for all real* t *and agreeing with* H *on* $[0, \pi/4]$, *satisfies* $\Lambda(t) = \Lambda(\pi/2 - t)$ *for all real* t. *This is a different extension from the one used in Chapter 12, which continues* E *past* $\pi/4$ *by the constant* $E(\pi/4)$; *the two disagree for* $t > \pi/4$.

Proof. (i) is Lemma 5.22. (ii) and (iii) are direct differentiation, using $\cos t \geq \sin t$ on $[0, \pi/4]$ with equality only at $t = \pi/4$, and $\cos t + \sin t \geq 1 > 0$ there. (iv) $\cos(\pi/4) + \sin(\pi/4) = \sqrt{2}$, and the bounds follow from (i) and (ii). (v) $\cos(\pi/2 - t) + \sin(\pi/2 - t) = \sin t + \cos t$; the two extensions already differ at $t = \pi/2$, where $\Lambda = 1$ while the constant extension of E gives $1 + E(\pi/4)$. $\square$

Chapter 12 extends E beyond $\pi/4$ by the constant value $E(\pi/4)$. Since $E'(\pi/4) = 0$, the extension is continuously differentiable; with concavity and $E(0) = 0$ this gives subadditivity of the extension, which together with the triangle inequality of Lemma 5.20(c) yields the frame-excess inequality $A + 1 \leq B + C$ used there.

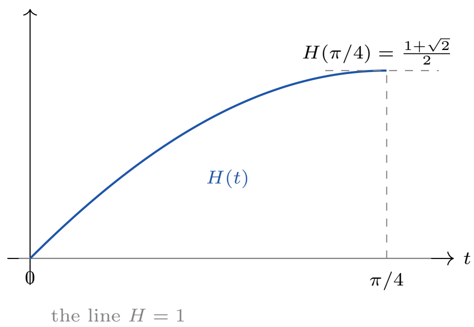


Figure 5.6: The threshold $H(t) = (1 + \cos t + \sin t)/2$ on $[0, \pi/4]$, drawn relative to the line $H = 1$ with the vertical scale exaggerated. The curve is increasing and strictly concave, with tangent of slope 0 at $t = \pi/4$.

Example 5.25 (The two owners of a branch give the same threshold). Let $F = (e_1, e_2)$ be the frame of a square centred at P and let G be the frame at angle $\pi/4$, with axis $m = (\cos\frac{\pi}{4}, \sin\frac{\pi}{4})$, of a square centred at P'. Then $d_4(F, G) = \pi/4$.

- Take $n = e_2$, an axis of F, so the first square owns the branch. Then $w_F(n) = \frac{1}{2}$ by Lemma 5.4(ii), and by (iii),

$$w_G(e_2) = \frac{|\langle e_2, m\rangle| + |\det(e_2, m)|}{2} = \tfrac{1}{2}\left(\tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2},$$

  so that $H = \frac{1+\sqrt{2}}{2}$.

- Take $n = m$, an axis of G, so the second square owns the branch. Then $w_G(m) = \frac{1}{2}$, while by Example 5.3,

$$w_F(m) = \frac{|\cos\frac{\pi}{4}| + |\sin\frac{\pi}{4}|}{2} = \frac{\sqrt{2}}{2}, \qquad H = \frac{1+\sqrt{2}}{2}.$$

Both equal $H(\pi/4)$, in agreement with Lemma 5.22.

Example 5.26 (The equality configuration)**.** Take three axis-parallel unit squares with centres

$$P_1 = (0,0), \qquad P_2 = (1,0), \qquad P_3 = (0,1).$$

By Example 5.15(a) the interiors are pairwise disjoint: the displacements are $(1,0)$, $(-1,1)$ and $(0,-1)$, each with a coordinate of absolute value 1. Indeed the first two squares meet only along the segment $x = \frac{1}{2}$, the first and third only along $y = \frac{1}{2}$, and the second and third only at the corner $(\frac{1}{2}, \frac{1}{2})$.

All three frames coincide, so every pairwise frame distance is 0 and every owned threshold is $H(0) = 1$. Orient the branches cyclically, with $d_i = P_{i+1} - P_i$:

$$\begin{array}{llll} d_1 = P_2 - P_1 = (1,0), & n_1 = e_1, & \langle d_1, n_1\rangle = 1, & H_1 = 1; \\ d_2 = P_3 - P_2 = (-1,1), & n_2 = e_2, & \langle d_2, n_2\rangle = 1, & H_2 = 1; \\ d_3 = P_1 - P_3 = (0,-1), & n_3 = -e_2, & \langle d_3, n_3\rangle = 1, & H_3 = 1. \end{array}$$

Every branch holds with equality, and every normal is an axis of every frame, so each branch is owned by both squares, and either may be designated the owner. The centre triangle has sides $1, 1, \sqrt{2}$, so it is right-angled and in particular non-obtuse, and its doubled area is

$$D = \sigma \det\big(P_2 - P_1,\ P_3 - P_1\big) = \sigma \det\big((1,0),(0,1)\big) = \sigma,$$

which is 1 for the orientation sign $\sigma = +1$; the area is $\frac{1}{2}$. Every threshold equals 1, so the branches are tight and the configuration attains area $\frac{1}{2}$. The constant $\frac{1}{2}$ in Theorem 1.1 is therefore sharp; see Chapter 13.

Exercises

Exercise 5.1. Let F be the axis-parallel frame and let G be the frame whose first vector has angle $\pi/6$. Compute $d_4(F,G)$; compute $w_G(n)$ for each of the four axes n of F, and $w_F(m)$ for each of the four axes m of G; and compute the threshold of an owned branch between two squares with these frames. Check the results against $H(\pi/6)$ and against Lemma 5.22.

Exercise 5.2. Thresholds in unowned directions.

(a) Let F be the axis-parallel frame, G the frame at angle $\pi/4$, and n the unit vector at angle $\pi/8$, which is an axis of neither. Show that

$$H(F,G;n) = \cos\tfrac{\pi}{8} + \sin\tfrac{\pi}{8} = \sqrt{1 + \tfrac{\sqrt{2}}{2}} \approx 1.3066,$$

and observe that this exceeds $H(\pi/4) = (1+\sqrt{2})/2 \approx 1.2071$. The owned directions are therefore not those of largest threshold.

(b) Show, using (C3), that $H(F,G;\cdot)$ is the support function h_Z of the obstacle $Z = Q(F,0) + Q(G,0)$.

(c) Prove that $H(F, G; n) \leq \sqrt{2}$ for all frames and all unit n, and determine exactly when equality holds. (Use $|a| + |b| \leq \sqrt{2}$ for $a^2 + b^2 = 1$.)

Exercise 5.3. Two axis-parallel unit squares have centres $P = (0, 0)$ and $P' = (1, \frac{1}{5})$, as in Example 5.11. Determine every unit vector n for which the branch (5.1) holds. (The answer is a single vector: the set of certifying directions can consist of one direction only.)

Exercise 5.4. Show that if $w_F(n) = w_G(n)$ for every unit vector n, then F and G have the same axis set, so that $\theta_F = \theta_G$ in the frame circle. (Hint: locate the minima of w_F using Lemma 5.4(iv).) Deduce that the map sending a frame class to the function w is injective on $\mathbb{T}$.

Exercise 5.5. Consider the three axis-parallel unit squares of Example 5.26. Show that no single unit vector n furnishes a branch, in one orientation or the other, for all three pairs. (Suggestion: a branch in direction $\pm n$ for the pair with displacement d requires $|\langle d, n\rangle| \geq H(F, G; n)$; test $d = (1, 0)$ and $d = (0, 1)$.) Conclude that pairwise disjointness of three squares requires three normals, so that no single direction replaces the tournament of Section 5.7.

Exercise 5.6. Exhibit two unit squares with disjoint interiors admitting two valid branches whose normals are neither parallel nor perpendicular, and identify the owner of each branch, if any. Explain why this does not contradict Lemma 5.18.

Exercise 5.7. Owner tournaments.

(a) Show that a tournament on three labelled vertices — one directed edge per pair — is, up to relabelling of the vertices, either *cyclic* (the three edges form a directed 3-cycle) or *transitive* (some vertex has out-degree 2, some has in-degree 2). Count how many of the eight tournaments are of each kind.

(b) For each of the two kinds, exhibit three unit squares with pairwise disjoint interiors together with a choice of owned branches whose owner tournament is of that kind.

(c) Deduce that Chapter 12 may legitimately assume, after relabelling, that the vertex of out-degree 2, the middle vertex and the vertex of in-degree 2 of a transitive owner tournament are P_1, P_2, P_3 in that order.

Exercise 5.8. Let F and G be frames at frame distance $t \in [0, \pi/4]$ and let $Z = Q(F, 0) + Q(G, 0)$. Using (C4) and (C5), show that for each axis direction n of F or of G the smallest $\lambda \geq 0$ for which the displacement $P' - P = \lambda n$ admits an owned branch is $\lambda = H(t)$. Interpret the monotonicity of H (Proposition 5.24(ii)): tilted squares need more room, and by $\max_{[0,\pi/4]} H = H(\pi/4)$ they need at most $\frac{\sqrt{2}-1}{2}$ more.

Chapter 6

Configuration Space and the Calculus of Area

Chapters 1–5 produced a geometric object: three centres, three unit normals, three thresholds and an orientation sign. This chapter puts that data into a form to which calculus applies. A triple of centres becomes a single point x of $\mathbb{R}^4$, each separating branch becomes a linear inequality with a constant gradient, and the doubled area becomes a quadratic function with a computable gradient. The norm of that gradient is bounded below in Section 6.6 and its pairing with x is evaluated in Section 6.7.

Assumed from earlier chapters.

- From Chapter 1: the doubled signed area $D = \sigma \det(P_2 - P_1, P_3 - P_1)$ of the centre triangle, and Area $= \frac{1}{2}|D|$.
- From Chapter 2: the quarter turn J_σ, together with $J_\sigma^2 = -I$ and $\langle J_\sigma u, v\rangle = \sigma \det(u, v)$; and the companion inequality $\|d\| \geq \langle d, n\rangle$ for a unit vector n.
- From Chapter 5: the oriented separating branch in the form $\langle d_i, n_i\rangle \geq H_i$, and the universal bound $H_i \geq 1$ for an owned branch.
- From Part IA: one-variable differentiation, Cauchy–Schwarz, and determinants of 2×2 matrices.

Notation for this chapter. In this chapter f denotes a generic real-valued function on $\mathbb{R}^n$, h a generic increment or vertex displacement, and z a translation of the plane; the four coordinates of the configuration are written x_1, x_2, x_3, x_4. The letters $n, d, P, H, \lambda, D, \sigma, J$ keep their global meanings. An *active* branch means one for which equality holds in the corresponding linear inequality; it does not refer to geometric tangency of squares, and no square is ever moved or rotated.

Standing hypothesis for this chapter. Nothing is frozen yet: the selection of a counterexample and of one owned branch per pair, and the declaration that the resulting data are constant, happen in Chapter 9. Every statement below that mentions normals or thresholds is therefore conditional, made under the hypothesis

$$\sigma \in \{-1, +1\}, \qquad n_1, n_2, n_3 \in \mathbb{R}^2 \text{ unit vectors,}$$
$$H_1, H_2, H_3 \in \mathbb{R} \text{ with } H_i \geq 1,$$

all fixed once and for all in what follows. Chapter 9 discharges the hypothesis by producing these data from a counterexample. Where a result needs in addition that a configuration satisfies the branch system $\langle d_i, n_i\rangle \geq H_i$, this is stated explicitly and called the *branch hypotheses*.

6.1 Gradients and directional derivatives

The only consequence of differentiability needed here is that a negative inner product with the gradient forces strict decrease along a ray.

Definition 6.1. A function $f : \mathbb{R}^n \to \mathbb{R}$ is *differentiable at* $x \in \mathbb{R}^n$ if there is a vector $g \in \mathbb{R}^n$ such that

$$f(x+h) = f(x) + \langle g, h\rangle + R(h) \qquad \text{for all } h \in \mathbb{R}^n,$$

where the remainder satisfies $R(h)/\|h\| \to 0$ as $h \to 0$. The vector g is then unique, and is written $g = \nabla f(x)$, the *gradient* of f at x.

For the uniqueness of g, suppose g and g' both work, and fix $v \in \mathbb{R}^n$ with $v \neq 0$. Taking $h = tv$ with $t > 0$ and subtracting the two expansions gives $\langle g - g', tv\rangle = R'(tv) - R(tv)$, so

$$\langle g - g', v\rangle = \frac{R'(tv) - R(tv)}{t} = \|v\| \cdot \frac{R'(tv) - R(tv)}{\|tv\|} \longrightarrow 0$$

as $t \downarrow 0$. The left-hand side does not depend on t, so it is 0; as v was arbitrary, $g = g'$.

Remark 6.2. The general differential calculus of several variables is not needed. Every function of the configuration that is differentiated in this book is a polynomial in the coordinates of x, and in each case the expansion of Definition 6.1 is exhibited directly, with R an explicit polynomial all of whose terms have degree at least 2 in h, so that $R(h)/\|h\| \to 0$ is immediate. No existence theorem, several-variable chain rule, or implication from continuous differentiability is used. Functions of a single real variable are differentiated freely and need not be polynomial; examples are the threshold function $H(t) = (1+\cos t+\sin t)/2$ of Chapter 5, the cyclic factor Φ of Chapter 11 and the certificate expressions in $r = \tan(t/2)$ of Chapter 12. These are one-variable arguments of the kind Part IA supplies.

Proposition 6.3. *Let f be differentiable at x and let $v \in \mathbb{R}^n$. Then $\varphi(t) = f(x+tv)$ is differentiable at $t = 0$, with $\varphi'(0) = \langle \nabla f(x), v\rangle$.*

Proof. If $v = 0$ then φ is constant and both sides are 0. If $v \neq 0$ then for $t \neq 0$,

$$\frac{\varphi(t) - \varphi(0)}{t} = \langle g, v\rangle + \frac{R(tv)}{t}, \qquad \left|\frac{R(tv)}{t}\right| = \|v\| \cdot \frac{|R(tv)|}{\|tv\|} \longrightarrow 0$$

as $t \to 0$, since $tv \to 0$. Hence the difference quotient tends to $\langle g, v\rangle$. $\square$

Lemma 6.4 (Strict descent)**.** *Let f be differentiable at x and let $v \in \mathbb{R}^n$ satisfy $\langle \nabla f(x), v\rangle < 0$. Then there is $\delta > 0$ such that*

$$f(x+tv) < f(x) \qquad \textit{for all } 0 < t < \delta.$$

Proof. Let $\varphi(t) = f(x+tv)$, so $\varphi'(0) = \langle \nabla f(x), v\rangle < 0$ by Proposition 6.3. Apply the definition of the limit $\lim_{t\to 0}\big(\varphi(t) - \varphi(0)\big)/t = \varphi'(0)$ with $\varepsilon = -\varphi'(0)/2 > 0$: there is $\delta > 0$ such that

$$\frac{\varphi(t) - \varphi(0)}{t} < \varphi'(0) + \varepsilon = \frac{\varphi'(0)}{2} < 0 \qquad \text{whenever } 0 < |t| < \delta.$$

For $0 < t < \delta$ multiply by $t > 0$ to get $\varphi(t) - \varphi(0) < t\varphi'(0)/2 < 0$. $\square$

The contrapositive is the first-order optimality condition: if x minimises f over a set that contains the segment $x + tv$ for all small $t > 0$, then $\langle \nabla f(x), v\rangle \geq 0$. It is used in Chapter 7 as the first-order condition, in Chapter 9 for motions of a single vertex, and in Chapter 10 along rays.

Example 6.5. Let $\sigma \in \{-1, +1\}$ and define $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by $f(x) = \sigma \det(P_2, P_3)$, where $x = (P_2, P_3)$. Write $h = (h_2, h_3)$. Since det is bilinear,

$$\begin{aligned} f(x+h) &= \sigma \det(P_2 + h_2,\ P_3 + h_3) \\ &= f(x) + \underbrace{\sigma \det(h_2, P_3) + \sigma \det(P_2, h_3)}_{\text{linear in } h} + \underbrace{\sigma \det(h_2, h_3)}_{=R(h)}. \end{aligned}$$

The first two terms are linear in h; the last satisfies $|R(h)| \leq \|h_2\|\,\|h_3\| \leq \|h\|^2$, so $R(h)/\|h\| \to 0$. Thus f is differentiable at every point, with remainder the explicit quadratic $\sigma \det(h_2, h_3)$. Its gradient is identified in Section 6.6.

6.2 Affine functions and the exact difference identity

Definition 6.6. A function $f : \mathbb{R}^n \to \mathbb{R}$ is *affine* if $f(x) = \langle g, x\rangle + c$ for some $g \in \mathbb{R}^n$ and $c \in \mathbb{R}$, and *linear* if in addition $c = 0$.

Lemma 6.7 (Affine calculus). *Let $f(x) = \langle g, x\rangle + c$ be affine on $\mathbb{R}^n$. Then:*

(i) f is differentiable at every point, with $\nabla f(x) = g$ for all x; the error term in Definition 6.1 is identically zero.

(ii) (Exact difference identity) *$f(x) - f(y) = \langle g, x - y\rangle$ for all $x, y \in \mathbb{R}^n$.*

(iii) $f(x + tv) = f(x) + t\langle g, v\rangle$ for all $x, v \in \mathbb{R}^n$ and all $t \in \mathbb{R}$.

(iv) $f(x - tg) = f(x) - t\|g\|^2$ for all $x \in \mathbb{R}^n$ and all $t \in \mathbb{R}$.

Proof. (i) $f(x + h) = \langle g, x\rangle + \langle g, h\rangle + c = f(x) + \langle g, h\rangle + 0$. (ii) Subtract $\langle g, y\rangle + c$ from $\langle g, x\rangle + c$ and use linearity of the inner product. (iii) Is (ii) with x replaced by $x + tv$ and y by x. (iv) Is (iii) with $v = -g$, since $\langle g, -g\rangle = -\|g\|^2$. □

By part (ii), the value of an affine function at two points determines the inner product of the gradient with their difference exactly, with no remainder and no restriction on the distance between the points. Chapter 9 uses this at global scale, evaluating the doubled area at the two vertices where it vanishes.

Part (iv) is the quantitative descent step: moving along $-g$ decreases an affine function at rate $\|g\|^2$ per unit of t, so a lower bound on $\|g\|$ bounds the rate of descent from below. This is why $\|g\|$ is computed in Section 6.6.

Example 6.8. Let f be affine on $\mathbb{R}^2$ with gradient $g \neq 0$, and suppose $f(P) = f(Q) = 0$ for two distinct points P, Q. By Lemma 6.7(ii), $\langle g, P - Q\rangle = 0$, so g is orthogonal to the line L through P and Q; and $f(X) = f(P) + \langle g, X - P\rangle = \langle g, X - P\rangle$ for every X. Write $X - P = \alpha\,\hat{g} + \beta\,\hat{u}$, where $\hat{g} = g/\|g\|$ and $\hat{u}$ is a unit vector along L; this is legitimate because $(\hat{g}, \hat{u})$ is an orthonormal basis of $\mathbb{R}^2$. Then $f(X) = \alpha\|g\|$.

The distance from X to L, meaning $\operatorname{dist}(X, L) = \inf\{\|X - Y\| : Y \in L\}$, is $|\alpha|$: a general point of L is $Y = P + \beta'\hat{u}$, and by orthonormality

$$\|X - Y\|^2 = \|\alpha\hat{g} + (\beta - \beta')\hat{u}\|^2 = \alpha^2 + (\beta - \beta')^2 \;\geq\; \alpha^2,$$

with equality exactly at $\beta' = \beta$, so the infimum is attained and equals $|\alpha|$. Hence

$$|f(X)| = \|g\| \cdot \operatorname{dist}(X, L),$$

and the level sets $\{f = \text{const}\}$ are the lines parallel to L, equally spaced with f changing by $\|g\|$ per unit of distance. (Exercise 6.3 proves the same thing in $\mathbb{R}^n$, by descent rather than by coordinates.)

In the complete-contact argument of Chapter 9, f is the doubled area as a function of one vertex, P and Q are the other two vertices, and the level line through the moving vertex is the locus of configurations of a given area.

6.3 Translation invariance and the normalisation $P_1 = 0$

Throughout, the cyclic edges of a triple (P_1, P_2, P_3) of points of $\mathbb{R}^2$ are

$$d_1 = P_2 - P_1, \qquad d_2 = P_3 - P_2, \qquad d_3 = P_1 - P_3, \qquad \text{so } d_1 + d_2 + d_3 = 0.$$

Lemma 6.9 (Translation invariance). *For $z \in \mathbb{R}^2$ let $T_z(P_1, P_2, P_3) = (P_1 + z, P_2 + z, P_3 + z)$. Then every cyclic edge d_i is unchanged by T_z; hence so is every branch value $\langle d_i, n_i\rangle$, every side length $\|d_i\|$, every angle of the triangle $P_1P_2P_3$, and the doubled area $D = \sigma\det(P_2 - P_1, P_3 - P_1)$.*

Proof. Each listed quantity is a function of the differences $P_j - P_k$ alone, and $(P_j + z) - (P_k + z) = P_j - P_k$. For the angles, recall from Chapter 2 that they are determined by the inner products of the edge vectors; for D, note $P_2 - P_1 = d_1$ and $P_3 - P_1 = -d_3$. □

Proposition 6.10 (Normalisation)**.** *The map*

$$\Pi : \mathbb{R}^6 \to \mathbb{R}^4, \qquad \Pi(P_1, P_2, P_3) = (P_2 - P_1,\ P_3 - P_1)$$

is surjective, and $\Pi(P) = \Pi(P')$ if and only if $P' = T_z(P)$ for some $z \in \mathbb{R}^2$: the fibres of Π are exactly the translation orbits. Moreover $\Pi(P_1, P_2, P_3) = \Pi(0, P_2 - P_1, P_3 - P_1)$, and by Lemma 6.9 the triple (P_1, P_2, P_3) and its normalised representative $(0, P_2 - P_1, P_3 - P_1)$ have the same branch values, side lengths, angles and doubled area D. Consequently a triple satisfies a given system of branch, angle and area constraints if and only if its normalised representative does, with the same value of D, and we may assume $P_1 = 0$ throughout.

Proof. Given $(u, v) \in \mathbb{R}^4$, the triple $(0, u, v)$ maps to it, so Π is surjective. If $P' = T_z(P)$ then $\Pi(P') = \Pi(P)$ by Lemma 6.9. Conversely, if $P_2' - P_1' = P_2 - P_1$ and $P_3' - P_1' = P_3 - P_1$, put $z = P_1' - P_1$; then $P_2' = P_2 + z$ and $P_3' = P_3 + z$, so $P' = T_z(P)$. The remaining assertions are Lemma 6.9 applied with $z = -P_1$. ☐

Remark 6.11. The normalisation is what makes the compactness argument of Chapter 7 possible. The set of triples in $\mathbb{R}^6$ satisfying the branch system is invariant under all translations T_z by Lemma 6.9, hence is unbounded whenever it is nonempty, and no minimiser could be extracted from it by the extreme value theorem. Passing to $\mathbb{R}^4$ quotients out this unboundedness.

With $P_1 = 0$ in force one may still vary the first vertex, renormalising afterwards. Later chapters argue by moving one vertex at a time, while velocities are recorded in $\mathbb{R}^4$; the following proposition translates between the two.

Proposition 6.12 (Translating back)**.** *Let $x = (P_2, P_3) \in \mathbb{R}^4$ be the normalised representative of a triple, so $P_1 = 0$. Let $h \in \mathbb{R}^2$.*

(i) Replacing P_2 by $P_2 + h$ and leaving the other vertices fixed replaces x by $x + (h, 0)$.
(ii) Replacing P_3 by $P_3 + h$ and leaving the other vertices fixed replaces x by $x + (0, h)$.
(iii) Replacing $P_1 = 0$ by h, leaving the other vertices fixed, and renormalising replaces x by $x + (-h, -h)$.

In tabular form:

motion of one vertex by $h \in \mathbb{R}^2$	*velocity $v \in \mathbb{R}^4$*
$P_2 \mapsto P_2 + h$	$v = (h, 0)$
$P_3 \mapsto P_3 + h$	$v = (0, h)$
$P_1 \mapsto P_1 + h$, *then renormalise*	$v = (-h, -h)$

Proof. (i) and (ii) are immediate, since x lists P_2 and P_3. For (iii), the new triple is (h, P_2, P_3), whose normalised representative is $\Pi(h, P_2, P_3) = (P_2 - h,\ P_3 - h) = x + (-h, -h)$. ☐

6.4 A triple of centres as a point of $\mathbb{R}^4$

Definition 6.13 (Configuration space)**.** With $P_1 = 0$, the *configuration* of a triple of centres is the single point

$$x = (P_2, P_3) \in \mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2,$$

and $\mathbb{R}^4$ carries the block inner product

$$\big\langle (u_1, u_2), (v_1, v_2) \big\rangle = \langle u_1, v_1 \rangle + \langle u_2, v_2 \rangle, \qquad \|(u_1, u_2)\|^2 = \|u_1\|^2 + \|u_2\|^2.$$

Remark 6.14 (Blockwise reading)*.* Under the identification $\mathbb{R}^4 \cong \mathbb{R}^2 \times \mathbb{R}^2$ obtained by concatenating coordinates, the block inner product is the standard inner product of $\mathbb{R}^4$: both are $\sum_{k=1}^{4} x_k y_k$. Hence an identity between vectors of $\mathbb{R}^4$ holds if and only if both of its planar components hold, and an inner product in $\mathbb{R}^4$ is the sum of two planar inner products. Both facts are used below without further comment; the first is how the multiplier equation of Chapter 10 is resolved into the reciprocal equations.

Proposition 6.15 (Branch functions)**.** *Under the standing hypothesis, define* $b_1, b_2, b_3 : \mathbb{R}^4 \to \mathbb{R}$, *where* $x = (P_2, P_3)$, *by*

$$b_1(x) = \langle P_2,\, n_1\rangle, \qquad b_2(x) = \langle P_3 - P_2,\, n_2\rangle, \qquad b_3(x) = \langle -P_3,\, n_3\rangle.$$

Then, with $P_1 = 0$, *we have* $b_i(x) = \langle d_i, n_i\rangle$ *for* $i = 1, 2, 3$, *so the branch system* $\langle d_i, n_i\rangle \geq H_i$ *reads*

$$b_i(x) \geq H_i \qquad (i = 1, 2, 3).$$

Each b_i *is linear on* $\mathbb{R}^4$ *(not merely affine), with constant gradient*

$$a_1 = (n_1, 0), \qquad a_2 = (-n_2, n_2), \qquad a_3 = (0, -n_3).$$

Proof. With $P_1 = 0$ we have $d_1 = P_2 - P_1 = P_2$, $d_2 = P_3 - P_2$ and $d_3 = P_1 - P_3 = -P_3$, which gives the three identifications $b_i(x) = \langle d_i, n_i\rangle$. For linearity, resolve into blocks using Remark 6.14:

$$b_1(x) = \langle P_2, n_1\rangle + \langle P_3, 0\rangle = \langle (n_1, 0), x\rangle,$$

$$b_2(x) = \langle P_2, -n_2\rangle + \langle P_3, n_2\rangle = \langle (-n_2, n_2), x\rangle,$$

$$b_3(x) = \langle P_2, 0\rangle + \langle P_3, -n_3\rangle = \langle (0, -n_3), x\rangle.$$

Each is of the form $\langle a_i, x\rangle$ with no constant term, hence linear, and Lemma 6.7(i) gives $\nabla b_i(x) = a_i$ at every x. □

Example 6.16 (The a_i are not unit vectors)**.** Since n_1, n_2, n_3 are unit vectors,

$$\|a_1\| = \|a_3\| = 1, \qquad \|a_2\|^2 = \|-n_2\|^2 + \|n_2\|^2 = 2, \quad \text{so} \quad \|a_2\| = \sqrt{2}.$$

The vector a_2 must not be replaced by $a_2/\sqrt{2}$. The multipliers λ_i produced in Chapter 10 are defined relative to these gradients through the equation $g = \sum_i \lambda_i a_i$; rescaling a_2 would rescale λ_2 by the reciprocal factor, and the sine system, in which λ_2 appears alongside λ_1 and λ_3 in equations of a fixed shape, would cease to hold. Record also, for later use, the pairings

$$\langle a_1, a_2\rangle = -\langle n_1, n_2\rangle, \qquad \langle a_2, a_3\rangle = -\langle n_2, n_3\rangle, \qquad \langle a_1, a_3\rangle = 0,$$

each read off blockwise; the last says that the branches numbered 1 and 3 constrain disjoint blocks of x.

Definition 6.17 (The objective)**.** With $P_1 = 0$, the *doubled area* is the function

$$D : \mathbb{R}^4 \to \mathbb{R}, \qquad D(x) = \sigma \det(P_2, P_3),$$

a homogeneous quadratic polynomial in the coordinates of x: writing $x = (x_1, x_2, x_3, x_4)$ with $P_2 = (x_1, x_2)$ and $P_3 = (x_3, x_4)$, we have $D = \sigma(x_1 x_4 - x_2 x_3)$.

We write D for both the function $D(P_1, P_2, P_3) = \sigma \det(P_2 - P_1, P_3 - P_1)$ of a triple, defined on $\mathbb{R}^6$ and used in Lemma 6.9 and Theorem 6.22, and the function $D(x)$ of the configuration just defined on $\mathbb{R}^4$. By Proposition 6.10 the two agree on the normalised representative, $D(x) = D(0, P_2, P_3)$.

Remark 6.18 (The objective is not convex). As a quadratic form, $D = \sigma(x_1 x_4 - x_2 x_3)$ decouples into the two independent blocks $\sigma x_1 x_4$ and $-\sigma x_2 x_3$, each of which is a hyperbolic form with eigenvalues $\pm\frac{1}{2}$. Hence the symmetric matrix of D has eigenvalues $+\frac{1}{2}$ and $-\frac{1}{2}$, each with multiplicity two, whatever the value of σ: the objective is indefinite. Convexity of the objective is therefore unavailable, which is why Chapters 7 and 8 develop first-order optimality and the Farkas lemma directly rather than quoting a theorem about convex minimisation.

6.5 Linear versus affine

Lemma 6.19. *For an affine function f on $\mathbb{R}^n$ the following are equivalent:*
(a) f is linear;
(b) $f(0) = 0$;
(c) $f(tx) = t\,f(x)$ for all $t \in \mathbb{R}$ and all $x \in \mathbb{R}^n$;
(d) $\langle \nabla f(x), x\rangle = f(x)$ for all $x \in \mathbb{R}^n$.

Proof. Write $f(x) = \langle g, x\rangle + c$; then $f(0) = c$, so (a) $\Leftrightarrow$ (b). If $c = 0$ then $f(tx) = \langle g, tx\rangle = tf(x)$, giving (a) $\Rightarrow$ (c); and (c) with $t = 0$ gives $f(0) = 0$, so (c) $\Rightarrow$ (b). Finally $\nabla f(x) = g$ by Lemma 6.7(i), so (d) reads $\langle g, x\rangle = \langle g, x\rangle + c$ for all x, i.e. $c = 0$; conversely $c = 0$ gives (d). □

Two later steps use the linearity clause of Proposition 6.15 rather than affineness alone, and both are instances of (d).

- **The strict dual vector** (Chapter 10). The Farkas lemma of Chapter 8 requires a vector w with $\langle a_i, w\rangle > 0$ for every constraint gradient a_i. The proof takes $w = x$, the configuration itself, giving $\langle a_i, x\rangle = b_i(x) = H_i \geq 1 > 0$. The first equality is (d), and it fails for an affine b_i that is not linear.

- **Euler's identity in degree** 1 (Section 6.7). The identity $\langle a_i, x\rangle = 1 \cdot b_i(x)$ converts the multiplier equation into $2D = \sum_i \lambda_i H_i$, which is used in Chapters 11 and 12.

Example 6.20 (What a shifted normalisation costs)**.** Suppose we had normalised the first centre to $P_1 = c$ for some fixed $c \neq 0$ rather than to $P_1 = 0$, and written the branch functions in the remaining variables $x = (P_2, P_3) \in \mathbb{R}^4$:

$$b_1(x) = \langle P_2 - c,\, n_1\rangle, \qquad b_2(x) = \langle P_3 - P_2,\, n_2\rangle, \qquad b_3(x) = \langle c - P_3,\, n_3\rangle.$$

The gradients are unchanged, $a_1 = (n_1, 0)$, $a_2 = (-n_2, n_2)$, $a_3 = (0, -n_3)$, since c is a constant. But now

$$\langle a_1, x\rangle = \langle P_2, n_1\rangle = b_1(x) + \langle c, n_1\rangle, \qquad \langle a_2, x\rangle = b_2(x),$$

$$\langle a_3, x\rangle = -\langle P_3, n_3\rangle = b_3(x) - \langle c, n_3\rangle.$$

Concretely, take $c = -(H_1 + 1)n_1$, so that $\langle c, n_1\rangle = -(H_1 + 1)$ because n_1 is a unit vector. Then a configuration with $b_1(x) = H_1$, that is, with the first branch active, has

$$\langle a_1, x\rangle = H_1 - (H_1 + 1) = -1 < 0,$$

whatever the admissible value of $H_1 \geq 1$, so the choice $w = x$ fails the Farkas hypothesis.

Of the three branch functions, b_2 is linear for every c, while b_1 is linear if and only if $\langle c, n_1\rangle = 0$ and b_3 is linear if and only if $\langle c, n_3\rangle = 0$; so, by Lemma 6.19, the set of translates making all three homogeneous is $\{c : \langle c, n_1\rangle = \langle c, n_3\rangle = 0\}$, which is $\{0\}$ if and only if n_1, n_3 are linearly independent, and is a line otherwise (Exercise 6.4). Since the standing hypothesis permits $n_1 = \pm n_3$, the branch functions alone do not force the normalisation; the objective does. With $P_1 = c$ the doubled area of the triple, as a function of x, is

$$D_c(x) = \sigma \det(P_2 - c,\, P_3 - c) = \sigma \det(P_2, P_3) - \sigma \det(P_2 - P_3,\, c),$$

by bilinearity and $\det(c, c) = 0$. Hence $D_c(tx) = t^2\sigma \det(P_2, P_3) - t\,\sigma \det(P_2 - P_3, c)$, while $t^2 D_c(x) = t^2\sigma \det(P_2, P_3) - t^2\sigma \det(P_2 - P_3, c)$; equating them gives $(t^2 - t)\det(P_2 - P_3, c) = 0$. If this holds for all t and all x then $\det(P_2 - P_3, c) = 0$ for all P_2, P_3, and since $P_2 - P_3$ ranges over all of $\mathbb{R}^2$ this forces $c = 0$. Thus $c = 0$ is the unique translate making the branch functions and the objective simultaneously homogeneous.

6.6 The area gradient is the quarter turn of the opposite edge

The gradient is computed first in the abstract, for the determinant, then for the doubled area at each vertex of the triangle, and finally in the $\mathbb{R}^4$ coordinates of Section 6.4.

Lemma 6.21 (Gradients of the determinant)**.** *Fix $\sigma \in \{-1, +1\}$ and set $\Delta(a, b) = \sigma \det(a, b)$ for $a, b \in \mathbb{R}^2$. For fixed b, the map $a \mapsto \Delta(a, b)$ is linear with gradient $-J_\sigma b$; for fixed a, the map $b \mapsto \Delta(a, b)$ is linear with gradient $J_\sigma a$.*

First proof, in coordinates. Take $\sigma = +1$; the general case multiplies everything by σ, and $J_{-1} = -J_{+1}$. With $a = (a_1, a_2)$ and $b = (b_1, b_2)$ we have $\det(a, b) = a_1 b_2 - a_2 b_1$, which is linear in a with gradient $(b_2, -b_1)$ and linear in b with gradient $(-a_2, a_1)$. Since $J_{+1}(u_1, u_2) = (-u_2, u_1)$, these are $-J_{+1}b$ and $J_{+1}a$ respectively. □

Second proof, invariant. By the Chapter 2 identity, $\Delta(a, b) = \langle J_\sigma a, b\rangle$, which is linear in b with gradient $J_\sigma a$ by Lemma 6.7(i). For the other slot we need the skewness of J_σ, and this follows from the identity alone: for all $u, v \in \mathbb{R}^2$,

$$\langle J_\sigma u, v\rangle = \sigma \det(u, v) = -\sigma \det(v, u) = -\langle J_\sigma v, u\rangle = -\langle u, J_\sigma v\rangle,$$

using antisymmetry of det in the second step and symmetry of the inner product in the last. (This is Proposition 2.8(iii).) Therefore $\Delta(a, b) = \langle J_\sigma a, b\rangle = \langle a, -J_\sigma b\rangle$, which is linear in a with gradient $-J_\sigma b$. □

The coordinate proof checks the signs; the invariant proof uses only the identity $\langle J_\sigma u, v\rangle = \sigma \det(u, v)$, and the same skewness computation recurs whenever J_σ is moved from one side of an inner product to the other.

The identity $J_\sigma^2 = -I$ gives the remaining property of J_σ needed here, that it is an isometry. By skewness and then $J_\sigma^2 = -I$,

$$\|J_\sigma u\|^2 = \langle J_\sigma u, J_\sigma u\rangle = -\langle u, J_\sigma^2 u\rangle = \langle u, u\rangle = \|u\|^2,$$

so $\|J_\sigma u\| = \|u\|$ for every u. This is the form in which Corollary 6.25 and Proposition 6.26 use it.

Theorem 6.22 (Vertex gradients)**.** *Let $D(P_1, P_2, P_3) = \sigma \det(P_2 - P_1, P_3 - P_1)$ and let $d_1 = P_2 - P_1$, $d_2 = P_3 - P_2$, $d_3 = P_1 - P_3$. When two of the three vertices are held fixed, D is an affine function of the remaining one, and the corresponding gradients are*

$$\nabla_{P_1} D = J_\sigma d_2, \qquad \nabla_{P_2} D = J_\sigma d_3, \qquad \nabla_{P_3} D = J_\sigma d_1.$$

In words, the gradient at a vertex is J_σ applied to the opposite edge.

Proof. The vertex P_3. With P_1, P_2 fixed, $D = \sigma \det(d_1,\ P_3 - P_1) = \langle J_\sigma d_1,\ P_3 - P_1\rangle$ by the Chapter 2 identity. As a function of P_3 this is $\langle J_\sigma d_1, P_3\rangle - \langle J_\sigma d_1, P_1\rangle$, affine with gradient $J_\sigma d_1$.

The vertex P_2. With P_1, P_3 fixed put $c = P_3 - P_1$, so $D = \sigma \det(P_2 - P_1,\ c) = \Delta(P_2 - P_1, c)$. By Lemma 6.21 this is affine in $P_2 - P_1$, hence in P_2, with gradient $-J_\sigma c = J_\sigma(P_1 - P_3) = J_\sigma d_3$.

The vertex P_1. With P_2, P_3 fixed, expand by bilinearity:

$$\begin{aligned}\sigma \det(P_2 - P_1,\ P_3 - P_1) = {} & \sigma \det(P_2, P_3) - \sigma \det(P_2, P_1) \\ & - \sigma \det(P_1, P_3) + \sigma \det(P_1, P_1),\end{aligned}$$

and the last term vanishes. Every surviving term is either constant in P_1 or linear in P_1, so D is affine in P_1; write $g_1 = \nabla_{P_1} D$ for its (constant) gradient. We identify g_1 by translation invariance; the direct computation from the coefficients is Exercise 6.5.

Fix $h \in \mathbb{R}^2$ and move the three vertices to $P_1 + h$, $P_2 + h$, $P_3 + h$ one at a time, applying the exact difference identity of Lemma 6.7(ii) at each step with the gradient evaluated at the current configuration.

- Moving P_1 to $P_1 + h$ changes D by $\langle g_1, h\rangle$.
- In the resulting configuration $(P_1 + h, P_2, P_3)$, the edge $P_1 - P_3$ has become $d_3 + h$, so by the case of P_2 already proved, moving P_2 to $P_2 + h$ changes D by $\langle J_\sigma(d_3 + h),\ h\rangle$.

- In the resulting configuration $(P_1 + h, P_2 + h, P_3)$, the edge $P_2 - P_1$ is again d_1, so by the case of P_3 already proved, moving P_3 to $P_3 + h$ changes D by $\langle J_\sigma d_1,\, h\rangle$.

The total change is 0 by Lemma 6.9. Moreover $\langle J_\sigma h, h\rangle = \sigma \det(h, h) = 0$, so the middle contribution is $\langle J_\sigma d_3, h\rangle$. Hence

$$0 = \big\langle g_1 + J_\sigma d_3 + J_\sigma d_1,\ h\big\rangle \qquad \text{for every } h \in \mathbb{R}^2,$$

which forces $g_1 = -J_\sigma(d_1 + d_3) = J_\sigma d_2$, using $d_1 + d_2 + d_3 = 0$.

Finally, the edge opposite P_1 is $P_2P_3 = d_2$, that opposite P_2 is $P_3P_1 = d_3$, and that opposite P_3 is $P_1P_2 = d_1$; so in each case the gradient is J_σ of the opposite edge. □

Corollary 6.23 (The three vertex gradients sum to zero)**.** $\nabla_{P_1} D + \nabla_{P_2} D + \nabla_{P_3} D = 0.$

Proof. $J_\sigma d_1 + J_\sigma d_2 + J_\sigma d_3 = J_\sigma(d_1 + d_2 + d_3) = J_\sigma 0 = 0$ by linearity of J_σ. □

Remark 6.24. Corollary 6.23 is translation invariance in differential form, and that is how it was used: the proof of Theorem 6.22 deduced the third gradient from the other two by requiring the sum to annihilate every diagonal displacement (h, h, h). Once the formulas are known the verification above suffices, but the fact behind it is translation invariance. The same dependence appears in Chapter 10, where the three reciprocal equations turn out to be dependent, their right-hand sides summing to zero.

Corollary 6.25 (Norm of the area gradient)**.** *For each i, $\|\nabla_{P_i} D\|$ equals the length of the side of the triangle opposite P_i, since J_σ is an isometry. If in addition the branch hypotheses hold, that is, $\langle d_i, n_i\rangle \geq H_i$ for each i with the unit normals n_i and thresholds $H_i \geq 1$ fixed by the standing hypothesis, then*

$$\|d_i\| \ \geq\ \langle d_i, n_i\rangle \ \geq\ H_i \ \geq\ 1,$$

so every vertex gradient has norm at least 1.

Proof. The first assertion is Theorem 6.22 together with $\|J_\sigma u\| = \|u\|$. For the second, the first inequality is the Cauchy–Schwarz companion $\|d\| \geq \langle d, n\rangle$ of Chapter 2, valid for any unit vector n; the second is the branch inequality; the third is the standing hypothesis $H_i \geq 1$. □

Combine this with Lemma 6.7(iv): moving a vertex P to $P - tg$, where $g = \nabla_P D$, changes the doubled area by $-t\|g\|^2 \leq -t$. Descent along $-g$ therefore reduces the doubled area at rate at least 1 per unit of t. This is used in the degree-0 and degree-1 cases of the complete-contact lemma of Chapter 9, and is where the bound $H \geq 1$ for an owned branch, from Chapter 5, enters the analysis.

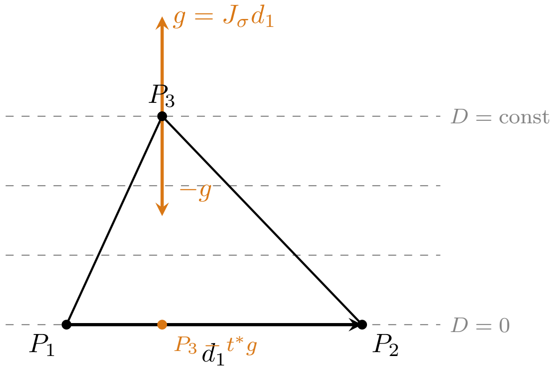


Figure 6.1: The area gradient at a vertex. With P_1, P_2 fixed, D is affine in P_3 with gradient $g = J_\sigma d_1$, perpendicular to d_1; its level lines (grey, dashed) are parallel to P_1P_2, and the line $D = 0$ is the line P_1P_2 itself. Since $\|g\| = \|d_1\|$, moving P_3 along $-g$ reduces D at rate $\|g\|^2 = \|d_1\|^2$, reaching $D = 0$ at $t^* = D/\|g\|^2$.

Proposition 6.26 (The objective gradient in $\mathbb{R}^4$)**.** *With $P_1 = 0$ and $D(x) = \sigma \det(P_2, P_3)$ as in Definition 6.17, D is differentiable at every $x \in \mathbb{R}^4$ with*

$$g := \nabla D(x) = \big(-J_\sigma P_3,\ J_\sigma P_2\big) = \big(J_\sigma d_3,\ J_\sigma d_1\big),$$

and $\|g\|^2 = \|d_3\|^2 + \|d_1\|^2$. Under the branch hypotheses, $\|g\|^2 \geq 2$.

Proof. Example 6.5 exhibits the expansion of Definition 6.1 with linear part $h \mapsto \sigma \det(h_2, P_3) + \sigma \det(P_2, h_3)$. By Lemma 6.21 the first summand is $\langle -J_\sigma P_3, h_2\rangle$ and the second is $\langle J_\sigma P_2, h_3\rangle$, so by Remark 6.14 the linear part is $\langle(-J_\sigma P_3, J_\sigma P_2), h\rangle$. Since $P_1 = 0$ we have $d_1 = P_2$ and $d_3 = -P_3$, whence $-J_\sigma P_3 = J_\sigma d_3$ and $J_\sigma P_2 = J_\sigma d_1$. The norm identity is the block formula of Definition 6.13 together with $\|J_\sigma u\| = \|u\|$, and the final bound is Corollary 6.25 applied to d_1 and d_3. □

The two blocks of g are $\nabla_{P_2} D$ and $\nabla_{P_3} D$ from Theorem 6.22: freezing P_3 and varying P_2 in $\mathbb{R}^4$ is the same motion as freezing P_1, P_3 and varying P_2 in the plane.

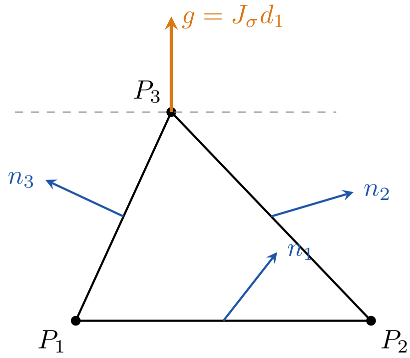


Figure 6.2: Branch normals and the area gradient. The normals n_1, n_2, n_3 (blue) are axes of frames fixed in advance; they bear no relation to the edges beside which they are drawn, and in particular are not perpendicular to them. The area gradient g (orange) is determined by the configuration and is perpendicular to the opposite edge.

Example 6.27 (A numerical check)**.** Take $\sigma = +1$, so $J_{+1}(u_1, u_2) = (-u_2, u_1)$, and

$$P_1 = (0, 0), \qquad P_2 = (2, 0), \qquad P_3 = (0, 1).$$

Then $D = \det\big((2, 0), (0, 1)\big) = 2$, and $d_1 = (2, 0)$, $d_2 = (-2, 1)$, $d_3 = (0, -1)$. Theorem 6.22 gives

$$\nabla_{P_1} D = J d_2 = (-1, -2), \qquad \nabla_{P_2} D = J d_3 = (1, 0), \qquad \nabla_{P_3} D = J d_1 = (0, 2).$$

Three checks. The three gradients sum to $(0, 0)$, as Corollary 6.23 requires. Their norms are $\sqrt{5}$, 1, 2, which are $\|d_2\|, \|d_3\|, \|d_1\|$, the lengths of the opposite sides, as Corollary 6.25 requires. Finally, move P_3 to $P_3 - tg$ with $g = (0, 2)$: by Lemma 6.7(iv),

$$D(t) = 2 - t\|g\|^2 = 2 - 4t,$$

which vanishes at $t^* = \frac{1}{2} = D/\|g\|^2$; and indeed $P_3 - \frac{1}{2}(0, 2) = (0, 0) = P_1$ lies on the line $P_1 P_2$, where the triangle degenerates, in agreement with Figure 6.1.

The example checks the algebra; the configuration is far from extremal, since $D = 2$ while the bound at issue is $D \geq 1$.

Example 6.28 (The $\mathbb{R}^4$ dictionary is consistent)**.** Let $x = (P_2, P_3)$ with $P_1 = 0$, and consider moving P_1 by h, which by Proposition 6.12(iii) is the velocity $v = (-h, -h) \in \mathbb{R}^4$. Using Proposition 6.26 and Remark 6.14,

$$\langle g, v\rangle = \langle J_\sigma d_3, -h\rangle + \langle J_\sigma d_1, -h\rangle = -\big\langle J_\sigma(d_1 + d_3),\, h\big\rangle = \langle J_\sigma d_2,\, h\rangle,$$

which is $\langle \nabla_{P_1} D, h\rangle$ by Theorem 6.22: the two descriptions of the same motion give the same derivative.

The same check for the branch functions. On the one hand,

$$\langle a_1, v\rangle = \langle n_1, -h\rangle = -\langle n_1, h\rangle, \qquad \langle a_2, v\rangle = \langle -n_2, -h\rangle + \langle n_2, -h\rangle = 0,$$
$$\langle a_3, v\rangle = \langle -n_3, -h\rangle = \langle n_3, h\rangle.$$

On the other hand, moving P_1 from 0 to h changes the edges to $d_1 - h$, d_2, $d_3 + h$, so the branch values become

$$\langle d_1, n_1\rangle - \langle h, n_1\rangle, \qquad \langle d_2, n_2\rangle, \qquad \langle d_3, n_3\rangle + \langle h, n_3\rangle,$$

in agreement. Chapter 9 works with single-vertex motions and Chapter 10 with $\mathbb{R}^4$ velocities; the computation above translates between them.

Example 6.29 (Preview of complete contact)**.** Fix P_1, P_2 and regard

$$D_{12}(X) = \sigma \det(P_2 - P_1,\ X - P_1)$$

as an affine function of the third vertex X, with gradient $g = J_\sigma d_1$ by Theorem 6.22. Then $D_{12}(P_1) = 0$, because $\det(u, 0) = 0$, and $D_{12}(P_2) = 0$, because $\det(u, u) = 0$. Hence by the exact difference identity, taking $X = P_3$,

$$\langle g,\ P_3 - P_1\rangle = D_{12}(P_3) - D_{12}(P_1) = D,$$
$$\langle g,\ P_3 - P_2\rangle = D_{12}(P_3) - D_{12}(P_2) = D.$$

The area gradient at P_3 pairs with the vector from either of the other two vertices to P_3 to give D, with no remainder and no restriction on the distance between the points. Combined with the half-plane duality of Chapter 9 in the form $g = \mu n$ with $\mu \geq 0$, this gives the active-degree-1 case of the complete-contact lemma. It is also Example 6.8 again, with $P = P_1$ and $Q = P_2$.

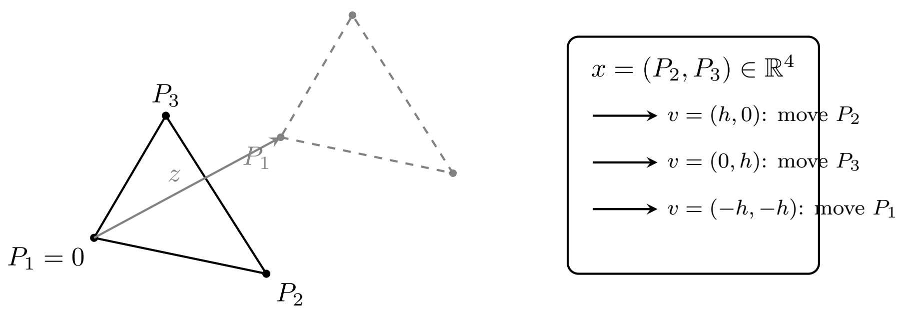


Figure 6.3: Each translation orbit is a single point of $\mathbb{R}^4$. The two triangles differ by the translation z (grey) and have the same edges, branch values, angles and doubled area; the normalised representative places P_1 at the origin. On the right, the configuration point of $\mathbb{R}^4$ together with the three velocities of Proposition 6.12.

6.7 Homogeneous functions and Euler's identity

Definition 6.30. A function $f : \mathbb{R}^n \to \mathbb{R}$ is *homogeneous of degree k* if $f(tx) = t^k f(x)$ for all $t > 0$ and all $x \in \mathbb{R}^n$.

Theorem 6.31 (Euler's identity)**.** *Let f be homogeneous of degree k and differentiable at x. Then*

$$\langle \nabla f(x),\ x\rangle = k\, f(x).$$

Proof. Define $\psi(t) = f(tx)$ for $t > 0$. On the one hand $\psi(t) = t^k f(x)$, so ψ is differentiable at $t = 1$ with $\psi'(1) = kf(x)$. On the other hand $\psi(1 + u) = f(x + ux)$, so by Proposition 6.3 applied at the point x in the direction $v = x$, the function $u \mapsto \psi(1 + u)$ is differentiable at $u = 0$ with derivative $\langle \nabla f(x), x\rangle$. The two computations give the same number. ∎

Corollary 6.32 (The two instances used)**.** *Under the standing hypothesis, with $P_1 = 0$ and $x = (P_2, P_3) \in \mathbb{R}^4$:*

(i) (Degree 2.) *$D(tx) = \sigma \det(tP_2, tP_3) = t^2 D(x)$, so $\langle g, x\rangle = 2D(x)$, where $g = \nabla D(x)$.*

(ii) (Degree 1.) *Each b_i is linear, hence $b_i(tx) = t\, b_i(x)$, so $\langle a_i, x\rangle = b_i(x)$ for $i = 1, 2, 3$.*

Proof. In (i), bilinearity of det gives $\det(tP_2, tP_3) = t^2 \det(P_2, P_3)$, so D is homogeneous of degree 2; it is differentiable by Proposition 6.26, and Theorem 6.31 applies with $k = 2$. In (ii), Proposition 6.15 says b_i is linear, so Lemma 6.19(c) gives homogeneity of degree 1 and Theorem 6.31 applies with $k = 1$; alternatively this is Lemma 6.19(d) directly. □

Part (ii) is where the normalisation $P_1 = 0$ is used. Without it, as in Example 6.20, the identity acquires the constants $\pm\langle c, n_i\rangle$; the branch functions are then affine but not homogeneous of any degree, and both later uses fail.

Example 6.33 (Verifying degree 2 by hand)**.** With $g = (J_\sigma d_3, J_\sigma d_1) = (-J_\sigma P_3,\ J_\sigma P_2)$ from Proposition 6.26,

$$\langle g, x\rangle = \langle -J_\sigma P_3,\ P_2\rangle + \langle J_\sigma P_2,\ P_3\rangle = -\sigma \det(P_3, P_2) + \sigma \det(P_2, P_3) = 2D,$$

using $\langle J_\sigma u, v\rangle = \sigma \det(u, v)$ twice and the antisymmetry $\det(v, u) = -\det(u, v)$. This confirms Corollary 6.32(i) without appeal to Theorem 6.31.

Proposition 6.34 (Euler bookkeeping)**.** *Under the standing hypothesis, suppose $x \in \mathbb{R}^4$ satisfies*

$$b_i(x) = H_i \quad (i = 1, 2, 3) \qquad \text{and} \qquad g = \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3$$

for some reals $\lambda_1, \lambda_2, \lambda_3$, where $g = \nabla D(x)$. Then

$$2D(x) = \langle g, x\rangle = \sum_{i=1}^{3} \lambda_i \langle a_i, x\rangle = \sum_{i=1}^{3} \lambda_i b_i(x) = \lambda_1 H_1 + \lambda_2 H_2 + \lambda_3 H_3.$$

Proof. The first equality is Corollary 6.32(i); the second is bilinearity of the inner product applied to the hypothesis $g = \sum_i \lambda_i a_i$; the third is Corollary 6.32(ii); the fourth is the hypothesis $b_i(x) = H_i$. □

Both hypotheses of Proposition 6.34 are supplied later. Chapters 7 and 9 produce a minimising configuration at which all three branches are active, which is the hypothesis $b_i(x) = H_i$. Chapter 8 proves the Farkas lemma and Chapter 10 applies it, producing the multiplier equation with the additional information $\lambda_i \geq 0$, which Proposition 6.34 does not use but the later chapters do. Chapters 11 and 12 then prove $\lambda_1 H_1 + \lambda_2 H_2 + \lambda_3 H_3 \geq 2$, whence $D \geq 1$.

Exercises

Exercise 6.1. Let $n \in \mathbb{R}^2$ be a unit vector. Compute the gradients on $\mathbb{R}^4$ of the functions

$$x \mapsto \langle P_3 - P_2,\ n\rangle \qquad \text{and} \qquad x \mapsto \|P_3 - P_2\|^2, \qquad x = (P_2, P_3),$$

in each case by exhibiting the expansion of Definition 6.1 explicitly. Show that the first function is linear and the second is not, but that the second is homogeneous of degree 2; verify Euler's identity for it directly, without quoting Theorem 6.31.

Exercise 6.2. Let $\sigma = +1$ and $P_1 = 0$, $P_2 = (1, 0)$, $P_3 = (\frac{1}{2}, 1)$.

(i) Compute $g = \nabla D(x) \in \mathbb{R}^4$ and $\|g\|$, and verify $\|g\|^2 = \|d_1\|^2 + \|d_3\|^2$.

(ii) Verify $\langle g, x\rangle = 2D$ numerically.

(iii) Let $v = (h, h)$ with $h \in \mathbb{R}^2$ arbitrary. Compute $\langle g, v\rangle$ and interpret the answer in terms of the vertex motion that v represents. (Hint: which row of the table in Proposition 6.12 is (h, h), up to sign?)

Exercise 6.3. Let f be affine on $\mathbb{R}^n$ with gradient $g \neq 0$, and let x satisfy $f(x) = c > 0$. Find the unique $t > 0$ with $f(x - tg) = 0$, and show that $x - tg$ is the point of the set $\{y : f(y) = 0\}$ nearest to x. Deduce that $\operatorname{dist}(x, \{f = 0\}) = c/\|g\|$, and reconcile this with Example 6.8.

Exercise 6.4. Suppose we normalise $P_1 = c$ with $c \neq 0$ rather than $P_1 = 0$, and write the branch functions and the objective in the variables $x = (P_2, P_3)$ as in Example 6.20.

(i) Determine the set of all $c \in \mathbb{R}^2$ for which b_1, b_2, b_3 are simultaneously homogeneous, and show that this set is $\{0\}$ if and only if n_1 and n_3 are linearly independent. Identify the set explicitly when $n_1 = \pm n_3$.

(ii) Exhibit c, n_1, n_3, thresholds $H_i \geq 1$ and a configuration x with $b_i(x) = H_i$ for all i but $\langle a_1, x\rangle < 0$, and conclude that with this normalisation the choice $w = x$ fails hypothesis (i) of the Farkas lemma.

(iii) Show, in contrast, that the objective D_c of Example 6.20 is homogeneous of degree 2 for $c = 0$ and, when $c \neq 0$, is homogeneous of no degree at all.

Exercise 6.5. Prove that $\nabla_{P_1} D = J_\sigma d_2$ directly: expand $\sigma \det(P_2 - P_1, P_3 - P_1)$ by bilinearity as a function of P_1 with P_2, P_3 held fixed, use the skewness of J_σ established in the second proof of Lemma 6.21 to write the linear part as an inner product, and read off the gradient. Do not use Corollary 6.23 or the translation argument of Theorem 6.22. Check your answer against Example 6.27.

Exercise 6.6. (The degree-0 case of complete contact, in the abstract.) Let f be affine on $\mathbb{R}^n$ with gradient $g \neq 0$, let $c_1, \ldots, c_m$ be affine functions, and let x satisfy $c_j(x) > 0$ for every j. Show that there is $\delta > 0$ such that for all $0 < t < \delta$,

$$f(x - tg) < f(x) \qquad \text{and} \qquad c_j(x - tg) > 0 \;\text{ for every } j.$$

Show further that $f(x) - f(x - tg) = t\|g\|^2 \geq t$ whenever $\|g\| \geq 1$. Where in your argument is the finiteness of m used, and what goes wrong for an infinite family?

Exercise 6.7. Let f be differentiable on $\mathbb{R}^n$ and homogeneous of degree k.

(i) Show that ∇f is homogeneous of degree $k - 1$, i.e. $\nabla f(tx) = t^{k-1}\nabla f(x)$ for $t > 0$. (Expand $f(tx + th)$ in two ways.)

(ii) Verify this for the objective D on $\mathbb{R}^4$.

(iii) Suppose x satisfies the branch system $b_i(x) \geq H_i$ for $i = 1, 2, 3$. Show that for $t > 0$ the dilated configuration tx (the centres dilated, the unit squares and the frozen data left alone) satisfies $b_i(tx) \geq tH_i$, and that $D(tx) = t^2 D(x)$; deduce that for $t \geq 1$ the branch system still holds, with the same thresholds. Explain why this produces neither a contradiction nor any new information about Theorem 1.1.

(iv) Now dilate squares and centres together by $t > 0$, so that the three squares have side t and the centres are tP_1, tP_2, tP_3. Identify the hypothesis of Theorem 1.1 that this destroys, and state and prove the correct scaled statement: if three squares of side t have pairwise disjoint interiors and their centre triangle is non-obtuse, then $\operatorname{Area}(P_1P_2P_3) \geq t^2/2$. (Reduce to the unit case by dividing all lengths by t, checking that disjointness of interiors and non-obtuseness are unaffected.)

Chapter 7

Compactness, Minima, and First-Order Optimality

The proof of Theorem 1.1 will proceed by choosing a counterexample that is as bad as possible: among all configurations that violate the conclusion, one whose centre triangle has the smallest doubled area. Two things must be established first: that such a configuration exists, since an infimum need not be attained; and that being a minimiser has usable local consequences, namely that no admissible motion of the centres decreases the area. This chapter proves both by ε–δ analysis.

What this chapter assumes

- **From Part IA.** The completeness axiom: every nonempty set of reals that is bounded below has an infimum. The algebra of limits for real sequences. The Bolzano–Weierstrass theorem on $\mathbb{R}$: every bounded real sequence has a convergent subsequence. The ε–δ definition of continuity for functions $\mathbb{R} \to \mathbb{R}$, and the continuity of $\sqrt{\cdot}$ on $[0, \infty)$.

- **From Chapter 1.** The open ball $B(x, \rho)$; the definitions of *open* and *closed* subsets of $\mathbb{R}^n$ and the interior $\operatorname{int} X$ of a set (Definition 1.2); the sequential characterisation of closedness (Proposition 1.7); the three equivalent forms of non-obtuseness of a triangle; and the fact that a non-obtuse triple of distinct points is not collinear (Lemma 1.28).

- **From Chapter 2.** The determinant $\det(u, v) = u_1v_2 - u_2v_1$ and its bilinearity; the Cauchy–Schwarz inequality and the determinant bound $|\det(u, v)| \le \|u\|\|v\|$, both corollaries of the Lagrange identity (Lemma 2.1); the working form $\langle d, n\rangle \le \|d\|$ for a unit vector n (Proposition 2.14(i)); and the subcritical triangle lemma, quoted in Example 7.21.

- **From Chapter 5.** The branch inequality $\langle d_i, n_i\rangle \ge H_i$ and the universal bound $H_i \ge 1$. These serve here only to name the constraints.

- **From Chapter 6.** The normalisation $P_1 = 0$ and the configuration point $x = (P_2, P_3) \in \mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$ with the block inner product; the linear branch functions

$$b_1(x) = \langle P_2, n_1\rangle, \qquad b_2(x) = \langle P_3 - P_2, n_2\rangle, \qquad b_3(x) = \langle -P_3, n_3\rangle$$

  with constant gradients $a_1 = (n_1, 0)$, $a_2 = (-n_2, n_2)$, $a_3 = (0, -n_3)$; the doubled area $D(x) = \sigma \det(P_2, P_3)$, a quadratic function with gradient $g = (-J_\sigma P_3,\ J_\sigma P_2)$; the vertex gradients $\nabla_{P_i} D$ and the bound $\|\nabla_{P_i} D\| \ge 1$ (Corollary 6.25); the Strict Descent Lemma 6.4, that $\langle \nabla f(x), v\rangle < 0$ forces $f(x + tv) < f(x)$ for all small $t > 0$; and, for affine f with gradient g, the identities $f(x) - f(y) = \langle g, x - y\rangle$ and $f(x - tg) = f(x) - t\|g\|^2$ (Lemma 6.7).

Sections 7.1–7.5 transport one-variable analysis to $\mathbb{R}^n$; the arguments are those of Part IA. Sections 7.6–7.8 are new.

Remark 7.1 (Notation for this chapter). In this chapter f denotes the objective of a minimisation problem, or, in §7.1 and §7.2, a generic real-valued function. The sets of active and slack constraint indices are written $\mathrm{Act}(x)$ and $\mathrm{Sl}(x)$, and the offset of an affine constraint is written θ. The number of constraints is N, and the letter k is always a sequence index.

7.1 Continuity in $\mathbb{R}^n$

Most of this section is the one-variable statement with $\|\cdot\|$ in place of $|\cdot|$, proved by the one-variable proof. The new item is the coordinatewise criterion, Lemma 7.3, and its blockwise form, Remark 7.4, which is the form used later.

Definition 7.2. A sequence $(x_k)_{k\geq 1}$ in $\mathbb{R}^n$ *converges* to $x \in \mathbb{R}^n$ if $\|x_k - x\| \to 0$ as $k \to \infty$; we write $x_k \to x$. Equivalently: for every $\varepsilon > 0$ there is K with $\|x_k - x\| < \varepsilon$ for all $k \geq K$.

Limits are unique: if $x_k \to x$ and $x_k \to y$ then $\|x-y\| \leq \|x-x_k\|+\|x_k-y\| \to 0$, so $\|x-y\| = 0$ and $x = y$. A subsequence of a convergent sequence converges to the same limit, by the same proof as in one variable.

Lemma 7.3 (Coordinatewise convergence)**.** *Write $x = (x^{(1)}, \dots, x^{(n)}) \in \mathbb{R}^n$. Then $x_k \to x$ in $\mathbb{R}^n$ if and only if $x_k^{(j)} \to x^{(j)}$ in $\mathbb{R}$ for each $j = 1, \dots, n$.*

Proof. For any $y \in \mathbb{R}^n$ and any j,

$$|y^{(j)}| \;\leq\; \|y\| \;=\; \Big(\textstyle\sum_i (y^{(i)})^2\Big)^{1/2} \;\leq\; \textstyle\sum_i |y^{(i)}| \;\leq\; n \max_i |y^{(i)}|. \tag{7.1}$$

The first inequality holds because $(y^{(j)})^2 \leq \sum_i (y^{(i)})^2$; the second because $\sum_i (y^{(i)})^2 \leq \big(\sum_i |y^{(i)}|\big)^2$, all cross terms being nonnegative. Apply (7.1) to $y = x_k - x$. The left-hand inequality gives $|x_k^{(j)} - x^{(j)}| \leq \|x_k - x\|$, so convergence in $\mathbb{R}^n$ implies convergence of each coordinate. The right-hand inequality gives $\|x_k - x\| \leq n \max_j |x_k^{(j)} - x^{(j)}|$, so if every coordinate converges then, given $\varepsilon > 0$, choosing K beyond which all n coordinates are within ε/n of their limits forces $\|x_k - x\| < \varepsilon$. ☐

Remark 7.4. In $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$ with the block inner product we have $\|(u_1, u_2)\|^2 = \|u_1\|^2 + \|u_2\|^2$, so the same two-sided estimate holds with blocks in place of coordinates:

$$\max\{\|u_1\|, \|u_2\|\} \;\leq\; \|(u_1, u_2)\| \;\leq\; \|u_1\| + \|u_2\|.$$

Hence $x_k = (P_{2,k}, P_{3,k}) \to (P_2, P_3)$ if and only if $P_{2,k} \to P_2$ and $P_{3,k} \to P_3$ in $\mathbb{R}^2$. This is the form in which Lemma 7.3 is used in Chapters 9 and 10.

Definition 7.5. Let $X \subseteq \mathbb{R}^n$ and $\varphi : X \to \mathbb{R}^m$. We say φ is *continuous at* $a \in X$ if for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$x \in X \text{ and } \|x - a\| < \delta \quad\Longrightarrow\quad \|\varphi(x) - \varphi(a)\| < \varepsilon.$$

We say φ is *continuous* if it is continuous at every point of X.

Proposition 7.6 (Sequential characterisation)**.** *With X, φ, a as above, φ is continuous at a if and only if $\varphi(x_k) \to \varphi(a)$ for every sequence (x_k) in X with $x_k \to a$.*

Proof. ($\Rightarrow$) Let $x_k \to a$ and let $\varepsilon > 0$. Take δ from continuity, then K with $\|x_k - a\| < \delta$ for $k \geq K$; for such k, $\|\varphi(x_k) - \varphi(a)\| < \varepsilon$.

($\Leftarrow$) By contraposition. Suppose φ is not continuous at a. Then there is $\varepsilon > 0$ for which no δ works; in particular $\delta = 1/k$ fails, so for each $k \geq 1$ there is $x_k \in X$ with $\|x_k - a\| < 1/k$ and $\|\varphi(x_k) - \varphi(a)\| \geq \varepsilon$. Then $x_k \to a$ but $\varphi(x_k) \not\to \varphi(a)$. ☐

The sequential form is what the proofs below use. The only ε–δ estimate remaining in the chapter is the one in the proof of Theorem 7.11(b),(c), from which the later persistence statements follow.

7.2 Polynomials, inner products and determinants are continuous

Proposition 7.7 (Algebra of continuity)**.** *Let $X \subseteq \mathbb{R}^n$ and $a \in X$.*

(i) If $\varphi, \psi : X \to \mathbb{R}$ are continuous at a and $\alpha \in \mathbb{R}$, then $\varphi + \psi$, $\alpha\varphi$ and $\varphi\psi$ are continuous at a.

(ii) A map $\varphi = (\varphi_1, \dots, \varphi_m) : X \to \mathbb{R}^m$ is continuous at a if and only if each component $\varphi_j : X \to \mathbb{R}$ is continuous at a.

(iii) If $\varphi : X \to \mathbb{R}^m$ is continuous at a, $\varphi(X) \subseteq Y \subseteq \mathbb{R}^m$, and $\Psi : Y \to \mathbb{R}^p$ is continuous at $\varphi(a)$, then $\Psi \circ \varphi$ is continuous at a.

Proof. All three are immediate from Proposition 7.6. For (i): if $x_k \to a$ then $\varphi(x_k) \to \varphi(a)$ and $\psi(x_k) \to \psi(a)$ in $\mathbb{R}$, so by the Part IA algebra of limits $\varphi(x_k) + \psi(x_k) \to \varphi(a) + \psi(a)$, and likewise for $\alpha\varphi$ and $\varphi\psi$. For (ii): if $x_k \to a$ then, by Lemma 7.3 applied in $\mathbb{R}^m$, $\varphi(x_k) \to \varphi(a)$ if and only if $\varphi_j(x_k) \to \varphi_j(a)$ for every j. For (iii): if $x_k \to a$ then $\varphi(x_k) \to \varphi(a)$, and $(\varphi(x_k))$ is a sequence in Y converging to $\varphi(a) \in Y$, so $\Psi(\varphi(x_k)) \to \Psi(\varphi(a))$. □

Proposition 7.8 (The basic continuous maps)**.** *(i) Each coordinate projection $\pi_j : \mathbb{R}^n \to \mathbb{R}$, $\pi_j(x) = x^{(j)}$, is continuous, as is each constant map.*

(ii) Every polynomial function $p : \mathbb{R}^n \to \mathbb{R}$ (a finite sum of finite products of coordinates and constants) is continuous.

(iii) On $\mathbb{R}^2 \times \mathbb{R}^2$ the maps $(u, v) \mapsto \langle u, v\rangle$, $(u, v) \mapsto \det(u, v)$ and $u \mapsto \|u\|^2$ are polynomial, hence continuous; and $u \mapsto \|u\|$ is continuous.

Proof. (i) By (7.1), $|\pi_j(x) - \pi_j(a)| = |x^{(j)} - a^{(j)}| \le \|x - a\|$, so $\delta = \varepsilon$ works. Constants are continuous with any δ.

(ii) A polynomial is obtained from the maps in (i) by finitely many additions, scalar multiplications and products, so Proposition 7.7(i) applies.

(iii) In coordinates $u = (u_1, u_2)$, $v = (v_1, v_2)$ we have $\langle u, v\rangle = u_1v_1 + u_2v_2$, $\det(u, v) = u_1v_2 - u_2v_1$ and $\|u\|^2 = u_1^2 + u_2^2$: all polynomial in the four variables u_1, u_2, v_1, v_2. Finally $\|u\| = \sqrt{\|u\|^2}$ is the composition of the continuous map $u \mapsto \|u\|^2$ with the continuous map $\sqrt{\cdot} : [0, \infty) \to \mathbb{R}$ from Part IA, so Proposition 7.7(iii) applies with $\varphi(u) = \|u\|^2$ and $\Psi = \sqrt{\cdot}$. □

Example 7.9 (Non-obtuseness as polynomial inequalities)**.** Fix the frozen unit normals n_1, n_2, n_3 and thresholds H_1, H_2, H_3 and the orientation sign σ, put $P_1 = 0$, and let $x = (P_2, P_3) \in \mathbb{R}^4$. The functions used in the rest of the book are the following.

function	formula in x	degree	continuous because
b_1	$\langle P_2, n_1\rangle$	1 (linear)	polynomial
b_2	$\langle P_3 - P_2, n_2\rangle$	1 (linear)	polynomial
b_3	$\langle -P_3, n_3\rangle$	1 (linear)	polynomial
D	$\sigma \det(P_2, P_3)$	2	polynomial
κ_1	$\langle P_2, P_3\rangle$	2	polynomial
κ_2	$\langle -P_2, P_3 - P_2\rangle = \|P_2\|^2 - \langle P_2, P_3\rangle$	2	polynomial
κ_3	$\langle -P_3, P_2 - P_3\rangle = \|P_3\|^2 - \langle P_2, P_3\rangle$	2	polynomial

Here κ_i is the inner product of the two edge vectors of the centre triangle emanating from P_i; each is a polynomial in the four coordinates of x, hence continuous by Proposition 7.8.

At the vertex P_1, the squared-side form of non-obtuseness from Chapter 1 says

$$\|P_2 - P_3\|^2 \le \|P_2 - P_1\|^2 + \|P_3 - P_1\|^2.$$

With $P_1 = 0$ the left-hand side expands as $\|P_2\|^2 - 2\langle P_2, P_3\rangle + \|P_3\|^2$, so the inequality is equivalent to $-2\langle P_2, P_3\rangle \le 0$, that is to $\kappa_1 \ge 0$. The same computation at P_2 (where the two edge vectors are $P_1 - P_2 = -P_2$ and $P_3 - P_2$) gives $\kappa_2 \ge 0$, and at P_3 it gives $\kappa_3 \ge 0$. Therefore "the centre triangle is non-obtuse" is the conjunction of the three polynomial inequalities $\kappa_1 \ge 0$, $\kappa_2 \ge 0$, $\kappa_3 \ge 0$, and "the centre triangle is strictly acute" is the conjunction of the three strict inequalities $\kappa_1 > 0$, $\kappa_2 > 0$, $\kappa_3 > 0$. This is the form in which non-obtuseness is used below.

A convention. The computation above assumed the three centres pairwise distinct, because Definition 1.26 of Chapter 1 defines "non-obtuse" only for a triple of pairwise distinct points: the angle at a vertex is undefined if two centres coincide. From this chapter onwards the constraint *non-obtuse* means $\kappa_1 \geq 0$, $\kappa_2 \geq 0$, $\kappa_3 \geq 0$, with no distinctness clause. On triples of distinct points this agrees with Definition 1.26, by the computation just given; on degenerate triples it is an extension, made so that the condition is closed. Without it closedness fails: the configurations $P_2 = (1, 0)$, $P_3 = (1, \varepsilon)$ have distinct centres and satisfy $\kappa_1 = 1$, $\kappa_2 = 0$, $\kappa_3 = \varepsilon^2$, all nonnegative, yet as $\varepsilon \downarrow 0$ they converge to the configuration $P_2 = P_3 = (1, 0)$, in which two centres coincide.

No such convention is needed for acuteness: $\kappa_i > 0$ for all i already forces the three centres to be distinct, since $P_2 = P_3$ gives $\kappa_2 = 0$, $P_2 = P_1 = 0$ gives $\kappa_1 = 0$, and $P_3 = P_1 = 0$ gives $\kappa_1 = 0$.

7.3 Open and closed conditions

Openness, closedness and the interior were defined in Chapter 1 (Definition 1.2): U is *open* if every point of U has a ball about it inside U, and X is *closed* if its complement is open. We quote from there, without proof, the characterisation used throughout this chapter.

Proposition 7.10 (Closed = sequentially closed; Proposition 1.7)**.** *$X \subseteq \mathbb{R}^n$ is closed if and only if the following holds: whenever (x_k) is a sequence in X and $x_k \to x$ in $\mathbb{R}^n$, we have $x \in X$.*

The proposition itself is not new; what is new is its range. In Chapter 1 the only sets whose closedness could be established were slabs, cut out by non-strict linear inequalities (Proposition 1.6), a linear inequality being the only kind whose stability had been proved. With the continuity results of §§7.1–7.2 we may replace "linear" by "continuous", and in particular by "polynomial", which is what the quadratic constraints $\kappa_i \geq 0$ and $0 \leq D \leq D_0$ need.

Theorem 7.11 (Open and closed conditions)**.** *Let $c_1, \dots, c_N : \mathbb{R}^n \to \mathbb{R}$ be continuous, with N finite.*

(a) $X = \{x \in \mathbb{R}^n : \ c_j(x) \geq 0$ for $j = 1, \dots, N\}$ is closed. The same holds for any set cut out by finitely many non-strict inequalities $\geq$, $\leq$ or equalities $=$ between continuous functions.

(b) $U = \{x \in \mathbb{R}^n : \ c_j(x) > 0$ for $j = 1, \dots, N\}$ is open.

(c) (Persistence.) *If $c_j(a) > 0$ for all j, then there is $\delta > 0$ such that $c_j(y) > 0$ for all j and all y with $\|y - a\| < \delta$. Consequently, for any $v \in \mathbb{R}^n$ there is $t_1 > 0$ with $c_j(a + tv) > 0$ for all j and all $|t| < t_1$.*

Proof. (a) Use Proposition 7.10. Let $(x_k) \subseteq X$ with $x_k \to x$. For each j, Proposition 7.6 gives $c_j(x_k) \to c_j(x)$, and $c_j(x_k) \geq 0$ for all k; since weak inequalities pass to limits of real sequences, $c_j(x) \geq 0$. Hence $x \in X$. The mixed case follows by replacing $c \leq 0$ with $-c \geq 0$ and $c = 0$ with the pair $c \geq 0$, $-c \geq 0$.

(b) and (c). Let a satisfy $c_j(a) > 0$ for all j, and set

$$\varepsilon \ = \ \min_{1 \leq j \leq N} c_j(a) \ > \ 0.$$

This is positive because the index set is finite. For each j, continuity of c_j at a gives $\delta_j > 0$ such that $\|y - a\| < \delta_j$ implies $|c_j(y) - c_j(a)| < \varepsilon$, whence $c_j(y) > c_j(a) - \varepsilon \geq 0$. Put $\delta = \min_j \delta_j > 0$, again positive because the list is finite. Then $\|y - a\| < \delta$ implies $c_j(y) > 0$ for every j. This proves (c), and proves (b) since $B(a, \delta) \subseteq U$ for every $a \in U$. For the last clause of (c), put $t_1 = \delta/(\|v\| + 1) > 0$; if $|t| < t_1$ then $\|(a + tv) - a\| = |t| \, \|v\| < \delta$. □

Remark 7.12 (Finiteness is necessary in (b)). With infinitely many strict inequalities, part (b) fails. On $\mathbb{R}$ take $c_k(x) = 1/k - x$ for $k = 1, 2, 3, \dots$; each c_k is continuous, but

$$\{x : \ c_k(x) > 0 \text{ for all } k\} = \{x : \ x < 1/k \text{ for all } k\} = (-\infty, 0],$$

which is not open, since every ball about 0 contains positive numbers. Every application of Theorem 7.11(b),(c) below is over a list of at most eight functions.

We record the consequence quoted in later chapters. Acuteness of the centre triangle ($\kappa_1, \kappa_2, \kappa_3 > 0$) and positivity of the doubled area ($D > 0$) are finite conjunctions of strict inequalities between continuous functions, hence open conditions: if they hold at a configuration x, they hold at every configuration near x, and in particular at $x + tv$ for every $v \in \mathbb{R}^4$ and all sufficiently small $|t|$.

Example 7.13 (The order of quantifiers in persistence)**.** The following instance shows what Theorem 7.11(c) does and does not give.

Take $\sigma = +1$, $P_1 = 0$, $P_2 = (1,0)$, $P_3 = (1,1)$. Then

$$\kappa_1 = \langle P_2, P_3\rangle = 1, \quad \kappa_2 = \|P_2\|^2 - \kappa_1 = 1 - 1 = 0, \quad \kappa_3 = \|P_3\|^2 - \kappa_1 = 2 - 1 = 1,$$

and $D = \det\big((1,0),(1,1)\big) = 1$. By Example 7.9 this triangle is non-obtuse but not acute: it has a right angle at P_2. Acuteness does not persist near this x; every ball about x contains configurations with $\kappa_2 < 0$, for instance $P_2 = (1,\eta)$, $P_3 = (1,1)$ with $\eta > 0$ small, for which $\kappa_2 = (1+\eta^2) - (1+\eta) = \eta^2 - \eta < 0$. So x lies on the boundary of the acute set, and Theorem 7.11(c) does not apply to it.

Now move to $P_2 = (1, -\frac{1}{10})$, $P_3 = (1,1)$. Then $\kappa_1 = 1 - \frac{1}{10} = \frac{9}{10}$, $\kappa_2 = (1 + \frac{1}{100}) - \frac{9}{10} = \frac{11}{100}$, $\kappa_3 = 2 - \frac{9}{10} = \frac{11}{10}$, all strictly positive, and $D = 1 + \frac{1}{10} > 0$. Here Theorem 7.11(c) applies and produces a $\delta > 0$ within which all four strict inequalities survive.

The δ produced by Theorem 7.11(c) depends on the point as well as on the list of functions; no single δ works at every configuration, and near a configuration where some κ_i vanishes there is none. This is why Chapter 9 proves that the minimiser is strictly acute, using the subcritical triangle lemma of Chapter 2, before appealing to the openness of acuteness.

7.4 Bolzano–Weierstrass in $\mathbb{R}^m$; compactness

Definition 7.14. A set $X \subseteq \mathbb{R}^n$ is *bounded* if $X \subseteq B(0,R)$ for some $R > 0$; a sequence is bounded if the set of its terms is bounded, i.e. if $\|x_k\| \le R$ for all k and some R.

Theorem 7.15 (Bolzano–Weierstrass in $\mathbb{R}^m$)**.** *Every bounded sequence in $\mathbb{R}^m$ has a convergent subsequence.*

Proof. Let (x_k) be a sequence in $\mathbb{R}^m$ with $\|x_k\| \le R$ for all k. By the left inequality in (7.1), each coordinate sequence satisfies $|x_k^{(j)}| \le \|x_k\| \le R$, so each of the m real sequences $(x_k^{(j)})_k$ is bounded.

We extract subsequences one coordinate at a time; since there are m coordinates, the process has a last step. Write $(x_k^{[0]})_k = (x_k)_k$. Suppose $1 \le j \le m$ and a subsequence $(x_k^{[j-1]})_k$ has been constructed along which the coordinates $1, \dots, j-1$ converge. The real sequence $\big((x_k^{[j-1]})^{(j)}\big)_k$ is bounded, so by Bolzano–Weierstrass on $\mathbb{R}$ it has a convergent subsequence; let $(x_k^{[j]})_k$ be the corresponding subsequence of $(x_k^{[j-1]})_k$. Along $(x_k^{[j]})_k$ the j-th coordinate converges by construction, and the coordinates $1, \dots, j-1$ still converge, to the same limits as before, because a subsequence of a convergent real sequence converges to the same limit.

We have $(x_k) \supseteq (x_k^{[1]}) \supseteq \cdots \supseteq (x_k^{[m]})$, and along $(x_k^{[m]})_k$ every one of the m coordinates converges, say $(x_k^{[m]})^{(j)} \to L^{(j)}$. By Lemma 7.3, $x_k^{[m]} \to L = (L^{(1)}, \dots, L^{(m)})$ in $\mathbb{R}^m$, and $(x_k^{[m]})$ is a subsequence of (x_k). □

Finiteness of m is essential: in infinite dimensions the coordinatewise extraction has no last step, and the theorem is false.

Definition 7.16. $X \subseteq \mathbb{R}^n$ is *sequentially compact* if every sequence in X has a subsequence converging to a limit lying in X.

Theorem 7.17. *If $X \subseteq \mathbb{R}^n$ is closed and bounded, then X is sequentially compact.*

Proof. Let $(x_k) \subseteq X$. Since X is bounded, (x_k) is a bounded sequence, so by Theorem 7.15 it has a subsequence $x_{k_i} \to x$ for some $x \in \mathbb{R}^n$. Since X is closed, Proposition 7.10 gives $x \in X$. □

Remark 7.18 (Convention). Throughout this book, *compact* abbreviates *closed and bounded subset of* $\mathbb{R}^n$, and the only property of compact sets ever used is the conclusion of Theorem 7.17. The converse of Theorem 7.17 is true and is Exercise 7.3 below. On subsets of $\mathbb{R}^n$ the various definitions of compactness agree; this is the Heine–Borel theorem, which is neither used nor proved here.

7.5 The extreme value theorem in $\mathbb{R}^n$

Theorem 7.19 (Attainment of the infimum)**.** *Let $X \subseteq \mathbb{R}^n$ be nonempty, closed and bounded, and let $f : X \to \mathbb{R}$ be continuous. Then f is bounded below on X and there exists $x_* \in X$ with $f(x_*) \leq f(x)$ for all $x \in X$. The corresponding statement for the supremum follows by applying this one to $-f$.*

Proof. Let $\mu = \inf f(X)$, where a priori $\mu \in \mathbb{R} \cup \{-\infty\}$: the set $f(X)$ is a nonempty set of reals, so it has an infimum in $\mathbb{R}$ if it is bounded below, and we write $\mu = -\infty$ otherwise. Nonemptiness of X makes $f(X)$ nonempty.

Choose a *minimising sequence* $(x_k) \subseteq X$ as follows: if $\mu \in \mathbb{R}$, then for each $k \geq 1$ the number $\mu + 1/k$ is not a lower bound for $f(X)$, so there is $x_k \in X$ with $\mu \leq f(x_k) < \mu + 1/k$; if $\mu = -\infty$, then $-k$ is not a lower bound, so there is $x_k \in X$ with $f(x_k) < -k$. In the first case $f(x_k) \to \mu$ in $\mathbb{R}$; in the second $f(x_k) \to -\infty$.

By boundedness of X and Theorem 7.15 there is a subsequence (x_{k_i}) converging to some $x_* \in \mathbb{R}^n$, and by closedness of X and Proposition 7.10 we have $x_* \in X$. By continuity of f and Proposition 7.6, $f(x_{k_i}) \to f(x_*) \in \mathbb{R}$.

A subsequence of $(f(x_k))$ has the same limiting behaviour as $(f(x_k))$ itself. In the case $\mu = -\infty$ this would force $f(x_{k_i}) \to -\infty$, contradicting convergence to the real number $f(x_*)$; so $\mu \in \mathbb{R}$ and f is bounded below. In the case $\mu \in \mathbb{R}$ we get $f(x_{k_i}) \to \mu$ and $f(x_{k_i}) \to f(x_*)$, so by uniqueness of limits $\mu = f(x_*)$. Since μ is a lower bound for $f(X)$, we conclude $f(x_*) \leq f(x)$ for all $x \in X$. □

Nonemptiness supplies the sequence, boundedness the convergent subsequence, closedness the limit point, and continuity the value.

Example 7.20 (Failure of each hypothesis)**.** Drop one hypothesis at a time and the conclusion fails.

(i) $X = (0, 1) \subseteq \mathbb{R}$ is bounded but not closed; $f(x) = x$ is continuous; $\inf f(X) = 0$ is not attained.

(ii) $X = [0, \infty)$ is closed but not bounded; $f(x) = 1/(1 + x^2)$ is continuous (a quotient of polynomials with nonvanishing denominator, or, if one prefers to stay inside Proposition 7.8, note that f is the composition of the polynomial $x \mapsto 1+x^2$ with the continuous map $u \mapsto 1/u$ on $(0, \infty)$); $\inf f(X) = 0$ is not attained.

(iii) $X = [0, 1]$ is closed and bounded; $f(x) = x$ for $x > 0$ and $f(0) = 1$ is not continuous at 0; $\inf f(X) = 0$ is not attained.

(iv) Over quadruples of admissible unit squares whose centres are in strictly convex position, the infimum of the area of the convex hull of the centres is 0, and it is not attained: this is Proposition 1.38, stated in Chapter 1 and proved in Chapter 13. The failure is not one of boundedness: the construction there uses axis-parallel unit squares in a row, with centres $(0, 0)$, (L, ε), $(2L, -\varepsilon)$, $(3L, 0)$, and one may fix $L = 2$ and let only $\varepsilon \downarrow 0$, so that every configuration in the minimising family lies inside one fixed bounded region. It is a failure of closedness: strict convex position is a conjunction of strict inequalities, hence an open condition by Theorem 7.11(b), and the limiting configuration at $\varepsilon = 0$ has its four centres collinear, so is not in the family.

Both points recur in Chapter 9, which secures closedness by writing every hypothesis of its frozen problem as a non-strict inequality ($\kappa_i \geq 0$ rather than $\kappa_i > 0$, and $0 \leq D \leq D_0$ rather than $0 < D < D_0$), so that Theorem 7.11(a) applies, and secures boundedness separately, by extracting an a priori bound on the side lengths of the centre triangle.

Example 7.21 (The existence lemma of Chapter 9)**.** This is the existence argument of Chapter 9, given here so that it may be cited there.

We take as given that every configuration $x = (P_2, P_3)$ satisfying the constraint list of Example 7.9 (that is: the three branch inequalities $b_i \geq H_i$, non-obtuseness $\kappa_i \geq 0$, and $0 \leq D \leq D_0 < 1$) has all three side lengths of its centre triangle in $[1, \sqrt{2})$. This is the subcritical triangle lemma of Chapter 2, applied to each branch inequality via the working form $\langle d_i, n_i\rangle \le \|d_i\|$ of Proposition 2.14(i).

Bounded. With $P_1 = 0$ the sides from P_1 are $\|P_2\|$ and $\|P_3\|$, so $\|P_2\| < \sqrt{2}$ and $\|P_3\| < \sqrt{2}$; by the block inner product (Remark 7.4),

$$\|x\|^2 = \|P_2\|^2 + \|P_3\|^2 < 2 + 2 = 4,$$

so $\|x\| < 2$; that is, the feasible set is contained in $B(0, 2) \subseteq \mathbb{R}^4$ and is bounded in the sense of Definition 7.14.

Closed. The feasible set is cut out by eight non-strict inequalities among the continuous functions of Example 7.9 (namely $b_i - H_i \ge 0$ for $i = 1, 2, 3$; $\kappa_i \ge 0$ for $i = 1, 2, 3$; $D \ge 0$; and $D_0 - D \ge 0$), so it is closed by Theorem 7.11(a).

Nonempty. By hypothesis: the assumed counterexample is a feasible point.

Conclusion. By Theorem 7.17 the feasible set is compact, and D is continuous on it, so by Theorem 7.19 there is a feasible x_* minimising D over the feasible set.

Geometry supplied boundedness; the remaining hypotheses came from the analysis of this chapter.

7.6 Constrained minimisation: the grammar

We fix the vocabulary and the sign conventions.

Definition 7.22 (Minimisation problem)**.** A *constrained minimisation problem* on $\mathbb{R}^n$ consists of an *objective* $f : \mathbb{R}^n \to \mathbb{R}$ together with a finite list of *constraint functions* $c_1, \ldots, c_N : \mathbb{R}^n \to \mathbb{R}$. Its *feasible set* is

$$X = \{y \in \mathbb{R}^n : \ c_j(y) \ge 0 \text{ for } j = 1, \ldots, N\},$$

and a point of X is a *feasible point.* A *global minimiser* is a point $x_* \in X$ with $f(x_*) \le f(y)$ for every $y \in X$. A *local minimiser* is a point $x_* \in X$ for which there is $\rho > 0$ with $f(x_*) \le f(y)$ for every $y \in X \cap B(x_*, \rho)$.

Remark 7.23 (All constraints in "≥ 0" form). Every inequality can be brought to this form, and here always is: $b(y) \ge H$ becomes $c(y) = b(y) - H \ge 0$; $f(y) \le \beta$ becomes $c(y) = \beta - f(y) \ge 0$; an equality $c(y) = 0$ is the pair $c \ge 0$, $-c \ge 0$. We do this silently from now on.

Remark 7.24. Every result in §7.7–§7.8 uses only that x_* beats nearby feasible points, so all of them hold for a local minimiser. We nevertheless produce a global minimiser in Chapter 9, since compactness supplies one.

Proposition 7.25 (Existence template)**.** *If f and all c_j are continuous on $\mathbb{R}^n$ and the feasible set X is nonempty and bounded, then a global minimiser exists.*

Proof. X is closed by Theorem 7.11(a) and bounded by hypothesis, hence compact; apply Theorem 7.19. □

Example 7.26 (The frozen problem)**.** Chapters 9, 10 and 13 all refer back to this list. With $P_1 = 0$, $n = 4$ and $x = (P_2, P_3)$:

Objective. $f(x) = D(x) = \sigma \det(P_2, P_3)$, the doubled signed area of the centre triangle.

Constraints (eight, all in "≥ 0" form):

$$\begin{aligned} &c_1(x) = \langle P_2, n_1\rangle - H_1, && c_2(x) = \langle P_3 - P_2, n_2\rangle - H_2, && c_3(x) = \langle -P_3, n_3\rangle - H_3, \\ &c_4(x) = \kappa_1, && c_5(x) = \kappa_2, && c_6(x) = \kappa_3, \\ &c_7(x) = D, && c_8(x) = D_0 - D. && \end{aligned}$$

The first three are the three frozen branches; each is affine, indeed a linear function minus a constant, with constant gradients

$$a_1 = (n_1, 0), \qquad a_2 = (-n_2, n_2), \qquad a_3 = (0, -n_3).$$

The next three express non-obtuseness at P_1, P_2, P_3 respectively (Example 7.9) and are quadratic. The last two are the sign constraint $D \geq 0$ and the objective bound $D \leq D_0$, and are quadratic.

Two points are used throughout §7.7–§7.8. First, c_1, c_2, c_3 are the only affine constraints on the list. Second, *active branch* means equality $c_i(x) = 0$ for some $i \in \{1, 2, 3\}$ in a chosen frozen linear inequality, not geometric tangency of squares: no square is moved or rotated, and the frames, normals, thresholds and orientation sign are constants.

Definition 7.27 (Active and slack)**.** Let x be a feasible point of the problem of Definition 7.22. Constraint j is *active at* x if $c_j(x) = 0$, and *slack at* x if $c_j(x) > 0$. Write $\mathrm{Act}(x) = \{j : c_j(x) = 0\}$ and $\mathrm{Sl}(x) = \{j : c_j(x) > 0\}$; since x is feasible these two sets partition $\{1, \dots, N\}$.

Example 7.28 (A fully computed configuration)**.** The example is used throughout §7.7–§7.8, and previews Chapter 10. One of its active constraints is not affine.

Take $\sigma = +1$, so that J_σ is the map $(a, b) \mapsto (-b, a)$. Take as frozen data

$$n_1 = (1, 0), \qquad n_2 = (0, 1), \qquad n_3 = -\tfrac{1}{\sqrt{2}}(1, 1), \qquad H_1 = H_2 = H_3 = 1,$$

and take the configuration $P_1 = 0$, $P_2 = (1, 0)$, $P_3 = (1, 1)$, with $D_0 = 1$.

The branch constraints.

$$\begin{aligned}
b_1 &= \langle P_2, n_1\rangle = \langle (1,0), (1,0)\rangle = 1, && \text{so } c_1 = 1 - 1 = 0: \ \textbf{active};\\
b_2 &= \langle P_3 - P_2, n_2\rangle = \langle (0,1), (0,1)\rangle = 1, && \text{so } c_2 = 1 - 1 = 0: \ \textbf{active};\\
b_3 &= \langle -P_3, n_3\rangle = \big\langle (-1,-1), -\tfrac{1}{\sqrt{2}}(1,1)\big\rangle = \sqrt{2}, && \text{so } c_3 = \sqrt{2} - 1 > 0: \ \text{slack}.
\end{aligned}$$

The shape constraints. $\kappa_1 = \langle P_2, P_3\rangle = 1$: slack. $\kappa_2 = \|P_2\|^2 - \langle P_2, P_3\rangle = 1 - 1 = 0$: **active** — the centre triangle has a right angle at P_2. $\kappa_3 = \|P_3\|^2 - \langle P_2, P_3\rangle = 2 - 1 = 1$: slack.

The area constraints. $D = \det\big((1,0), (1,1)\big) = 1$, so $c_7 = 1$ is slack and $c_8 = D_0 - D = 0$ is **active**.

The objective gradient. By Chapter 6, $g = (-J_\sigma P_3,\ J_\sigma P_2)$. Here $J_\sigma P_3 = (-1, 1)$ and $J_\sigma P_2 = (0, 1)$, so

$$g = \big((1, -1),\ (0, 1)\big).$$

Euler's identity for the quadratic D says $\langle g, x\rangle = 2D$; check it: $\langle g, x\rangle = \langle (1,-1), (1,0)\rangle + \langle (0,1), (1,1)\rangle = 1 + 1 = 2 = 2D$, as claimed.

The active list at this configuration contains a non-affine constraint ($c_5 = \kappa_2$) and an objective-bound constraint (c_8); §7.7 treats each. And one checks directly, using $a_1 = \big((1,0), (0,0)\big)$, $a_2 = \big((0,-1), (0,1)\big)$, $a_3 = \big((0,0), \tfrac{1}{\sqrt{2}}(1,1)\big)$, that

$$g = 1 \cdot a_1 + 1 \cdot a_2 + 0 \cdot a_3,$$

so nonnegative multipliers $(\lambda_1, \lambda_2, \lambda_3) = (1, 1, 0)$ exist, the slack branch receives multiplier 0, and $\sum_i \lambda_i H_i = 1 + 1 + 0 = 2 = 2D$, an instance of the Euler identity of Chapter 10.

The data here are chosen for arithmetic convenience: the thresholds $H_1 = H_2 = H_3 = 1$ are not those of any pair of frames, the normals come from no ownership argument, and $D_0 = 1$, whereas $D_0 < 1$ elsewhere in the book. In particular this configuration, having $D = 1$, is not feasible for the frozen problem of Chapter 9.

7.7 Perturbing a minimiser: which constraints can break

Suppose x_* is feasible and we push it with velocity v, considering the moving point $x(t) = x_* + tv$ for small $t > 0$. Three cases arise, treated in the three lemmas below: a constraint may be slack, active and affine, or an objective bound.

Lemma 7.29 (Slack constraints impose no local restriction)**.** *Let c be continuous at x with $c(x) > 0$. Then there is $\delta > 0$ with $c(y) > 0$ for all $y \in B(x, \delta)$; hence for every $v \in \mathbb{R}^n$ there is $t_1 > 0$ with $c(x + tv) > 0$ for all $|t| < t_1$. If c_j, $j \in \mathrm{Sl}(x)$, is a finite family of constraints all slack at x, one δ, and one t_1, works for all of them simultaneously.*

Proof. Apply Theorem 7.11(c) to the one-element list $\{c\}$, and then to the finite list $\{c_j\}_{j\in \mathrm{Sl}(x)}$. □

At the minimiser, acuteness and $D > 0$ are slack constraints, and Lemma 7.29 is the precise form of the statement that they are open conditions persisting for small t: the open-condition language of §7.3 and the slack language of §7.6 describe the same fact.

Lemma 7.30 (Active affine constraints and half-spaces)**.** *Let $c(y) = \langle a, y\rangle - \theta$ be affine and active at x, i.e. $\langle a, x\rangle = \theta$. Then for every $v \in \mathbb{R}^n$ and every $t \in \mathbb{R}$,*

$$c(x + tv) = \langle a, x\rangle + t\langle a, v\rangle - \theta = t\,\langle a, v\rangle.$$

Consequently:

(i) if $\langle a, v\rangle \geq 0$ then $c(x + tv) \geq 0$ for all $t \geq 0$, so the constraint survives along the whole ray and not only for small t;

(ii) if $\langle a, v\rangle < 0$ then $c(x + tv) < 0$ for every $t > 0$, so the constraint fails immediately.

So an active affine constraint confines admissible velocities to the closed half-space $\{v : \langle a, v\rangle \geq 0\}$.

Proof. The displayed identity is the linearity of $y \mapsto \langle a, y\rangle$ together with $\langle a, x\rangle = \theta$. Clauses (i) and (ii) read off the sign of $t\langle a, v\rangle$. □

Clause (i) is what Chapter 10 uses to apply the Farkas lemma along whole rays rather than short segments. It is available because the normalisation $P_1 = 0$ of Chapter 6 makes the branch functions b_i linear, so that $c_i = b_i - H_i$ is affine with constant gradient a_i.

Lemma 7.31 (Objective-bound constraints never obstruct descent)**.** *Let $c(y) = \beta - f(y)$ where f is the objective, and suppose $c(x) \geq 0$. If y is any point with $f(y) \leq f(x)$, then*

$$c(y) = \beta - f(y) \;\geq\; \beta - f(x) = c(x) \;\geq\; 0.$$

In particular, along any motion on which f does not increase, the constraint $c \geq 0$ is preserved, whether or not it was active at x.

Proof. Immediate from $f(y) \leq f(x)$. □

The constraint $c_8 = D_0 - D$ of Example 7.26 is quadratic and may be active, so neither Lemma 7.29 nor Lemma 7.30 applies to it; Lemma 7.31 does, and this is why c_8 contributes no multiplier in Chapter 10.

Remark 7.32 (Active non-affine constraints). For a differentiable constraint c active at x, the condition $\langle \nabla c(x), v\rangle \geq 0$ does not imply $c(x + tv) \geq 0$ for small $t > 0$. Take $n = 1$, $c(y) = -y^2$, $x = 0$, $v = 1$: then $\nabla c(0) = 0$, so $\langle \nabla c(0), v\rangle = 0 \geq 0$, yet $c(x + tv) = -t^2 < 0$ for every $t \neq 0$. The strict inequality $\langle \nabla c(x), v\rangle > 0$ does suffice, directly from the definition of the directional derivative, but strictness is not available here.

Theorem 7.36 therefore requires that every active constraint be affine, an objective bound, or slack. Chapter 9 proves that the minimiser is strictly acute with $D > 0$ so that $c_4, \ldots, c_7$ are slack there, leaving only the affine branch constraints active.

Remark 7.33 (Motions that move one vertex; the translation normalisation). In $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$, a velocity $v = (v_2, 0)$ moves P_2 alone and $v = (0, v_3)$ moves P_3 alone. Moving P_1 alone is legitimate too, despite the normalisation $P_1 = 0$: replace P_1 by $P_1 + tv_1$ and then translate all three centres by $-tv_1$ to restore $P_1 = 0$. Since every c_j and the objective f depend only on differences of the centres (Chapter 6), this re-normalisation changes neither feasibility nor the objective value. In the coordinates $x = (P_2, P_3)$ the resulting velocity is

$$v = (-v_1,\, -v_1).$$

Chapter 9 uses all three single-vertex motions.

7.8 First-order optimality: no feasible descent direction

Definition 7.34 (Admissible velocities)**.** Let x_* be a feasible point, and suppose that every constraint active at x_* is either affine, $c_j(y) = \langle a_j, y\rangle - \theta_j$, or of objective-bound type $c_j(y) = \beta_j - f(y)$ as in Lemma 7.31. Split the active set accordingly:

$$\mathrm{Aff}(x_*) = \{j \in \mathrm{Act}(x_*) : c_j \text{ affine}\}, \qquad \mathrm{Obj}(x_*) = \{j \in \mathrm{Act}(x_*) : c_j = \beta_j - f\},$$

so that $\mathrm{Aff}(x_*)$, $\mathrm{Obj}(x_*)$ and $\mathrm{Sl}(x_*)$ partition $\{1, \dots, N\}$. (A constraint of both shapes may be assigned to either set; the statements below do not depend on the choice.) Note that $c_j(x_*) = 0$ for $j \in \mathrm{Obj}(x_*)$ means $\beta_j = f(x_*)$. Set

$$V(x_*) = \{v \in \mathbb{R}^n : \ \langle a_j, v\rangle \geq 0 \text{ for every } j \in \mathrm{Aff}(x_*)\}.$$

Then $0 \in V(x_*)$; $V(x_*)$ is closed under multiplication by nonnegative scalars and under addition; and $V(x_*) = \mathbb{R}^n$ when $\mathrm{Aff}(x_*) = \emptyset$. Sets of this shape are the *cones* of Chapter 8; nothing about them beyond the displayed definition is needed here.

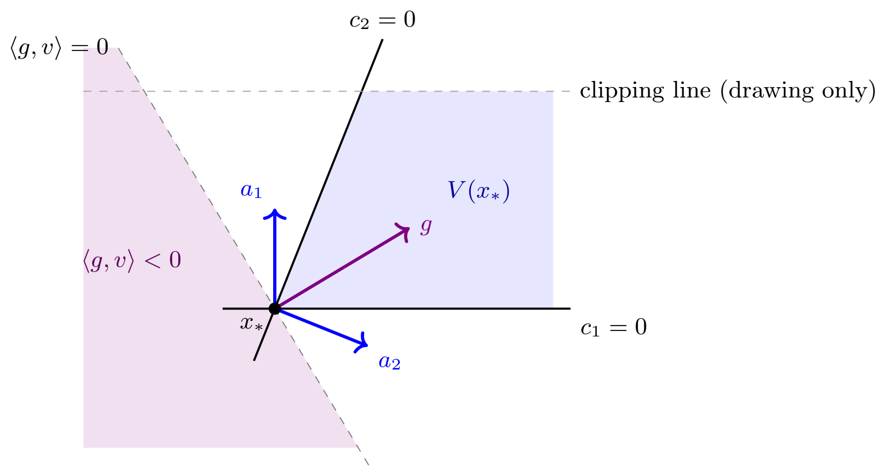


Figure 7.1: First-order optimality in the plane, with $\mathbb{R}^2$ standing in for $\mathbb{R}^4$. At the feasible point x_* the constraints c_1 and c_2 are affine and active, so $\mathrm{Aff}(x_*) = \{1, 2\}$; their normals a_1, a_2 are drawn in blue and the shaded blue region is the cone $V(x_*)$, unbounded and drawn clipped by the grey dashed line to fit the page. Any further constraints are slack at x_* and do not restrict $V(x_*)$, so none is drawn. The shaded region beyond the dashed grey line $\langle g, v\rangle = 0$ is the open descent half-plane $\langle g, v\rangle < 0$; Theorem 7.36 asserts that it does not meet the shaded cone.

Definition 7.35 (Descent direction)**.** If f is differentiable at x_* with gradient g, a vector $v \in \mathbb{R}^n$ is a *descent direction* at x_* if $\langle g, v\rangle < 0$.

Theorem 7.36 (First-order optimality: no feasible descent direction)**.** *Let x_* be a global, or merely local, minimiser of f over $X = \{y : c_j(y) \geq 0, \ j = 1, \dots, N\}$, and suppose:*

(a) f is differentiable at x_* with gradient g;

(b) every constraint slack at x_* is continuous at x_*;

(c) every constraint active at x_* is either affine, $c_j(y) = \langle a_j, y\rangle - \theta_j$ for $j \in \mathrm{Aff}(x_*)$, or of objective-bound type $c_j(y) = \beta_j - f(y)$ for $j \in \mathrm{Obj}(x_*)$, in which case $\beta_j = f(x_*)$ because $c_j(x_*) = 0$.

Then

$$\langle g, v\rangle \ \geq \ 0 \qquad \textit{for every } v \in V(x_*).$$

Proof. Suppose, for a contradiction, that some $v \in V(x_*)$ has $\langle g, v\rangle < 0$. Write $x(t) = x_* + tv$ and $\varphi(t) = f(x(t))$.

The objective strictly decreases. By hypothesis (a) and the Strict Descent Lemma 6.4 of Chapter 6, applied with $\varphi'(0) = \langle g, v\rangle < 0$, there is $t_0 > 0$ such that

$$f(x(t)) = \varphi(t) < \varphi(0) = f(x_*) \qquad \text{for all } 0 < t \le t_0. \tag{7.2}$$

The active affine constraints survive. Let $j \in \mathrm{Aff}(x_*)$. Since $v \in V(x_*)$ we have $\langle a_j, v\rangle \ge 0$, so by Lemma 7.30(i),

$$c_j(x(t)) = t\,\langle a_j, v\rangle \;\ge\; 0 \qquad \text{for all } t \ge 0.$$

The active objective bounds survive. Let $j \in \mathrm{Obj}(x_*)$, so $c_j(y) = \beta_j - f(y)$ with $\beta_j = f(x_*)$. For $0 < t \le t_0$, (7.2) and Lemma 7.31 give

$$c_j(x(t)) = \beta_j - f(x(t)) \;>\; \beta_j - f(x_*) \;=\; 0.$$

The slack constraints survive for small t. The set $\mathrm{Sl}(x_*)$ of slack indices is finite and each c_j, $j \in \mathrm{Sl}(x_*)$, is continuous at x_* with $c_j(x_*) > 0$, so by Lemma 7.29 there is a single $t_1 > 0$ with $c_j(x(t)) > 0$ for all $j \in \mathrm{Sl}(x_*)$ and all $|t| < t_1$.

Conclusion. Choose any t with $0 < t < \min(t_0, t_1)$; if x_* is only a local minimiser with radius ρ, shrink t further so that $\|x(t) - x_*\| = t\|v\| < \rho$. By Definition 7.34 every constraint index lies in $\mathrm{Aff}(x_*)$, $\mathrm{Obj}(x_*)$ or $\mathrm{Sl}(x_*)$, and we have just checked $c_j(x(t)) \ge 0$ in each case, so $x(t) \in X$; and $f(x(t)) < f(x_*)$ by (7.2). This contradicts the minimality of x_*. □

Corollary 7.37 (Whole-ray version)**.** *Let x_* be a feasible point (minimality is not assumed), and suppose every constraint is either affine and active at x_*, or of objective-bound type. Let $v \in \mathbb{R}^n$ satisfy $\langle a_j, v\rangle \ge 0$ for every $j \in \mathrm{Aff}(x_*)$, and suppose $f(x_* + tv) \le f(x_*)$ for all $t \ge 0$. Then $x_* + tv \in X$ for every $t \ge 0$.*

Proof. The constraints in $\mathrm{Aff}(x_*)$ hold for all $t \ge 0$ by Lemma 7.30(i), and the objective bounds hold for all $t \ge 0$ by Lemma 7.31. □

The second hypothesis on v is a statement about f along the whole ray, not about the sign of $\langle g, v\rangle$: for a non-affine f the sign of $\langle g, v\rangle$ controls f only for small t, and the objective D is quadratic.

The hypothesis of Corollary 7.37, that every constraint is affine-active or an objective bound, is not met by the problem of Example 7.26, since the acuteness constraints are quadratic and slack at the minimiser. Chapter 10 uses the mixed form appearing in the proof of Theorem 7.36: the three branch constraints are active and affine, so they hold along the whole ray by Lemma 7.30(i), while the remaining constraints are slack and need only small t.

Corollary 7.38 (A single active constraint)**.** *If $\mathrm{Aff}(x_*) = \{j\}$ consists of a single index with $a_j \ne 0$, then $V(x_*)$ is the closed half-space $\{v : \langle a_j, v\rangle \ge 0\}$, and Theorem 7.36 says $\langle g, v\rangle \ge 0$ for every v in that half-space.*

From this one may conclude that $g = \mu a_j$ for some $\mu \ge 0$; that is the half-plane lemma of Chapter 8, and we do not prove it here. (It is Exercise 7.7.)

Remark 7.39 (Why inactive constraints contribute nothing)*.* In the proof, a slack constraint is used only to fix t_1, and an objective-bound constraint only through Lemma 7.31; neither appears in the conclusion. Only the affine-active constraints do, through the definition of $V(x_*)$. Hence the multipliers of Chapter 10 are indexed by the three branches, and a branch that happened to be slack would receive multiplier 0.

Theorem 7.36 is the geometric half of what is standardly called the Karush–Kuhn–Tucker conditions; the other half, the production of multipliers, is Chapter 8.

Remark 7.40 (The bookkeeping checklist)*.* Given a minimiser x_* and a proposed velocity v, to certify that $x_* + tv$ is feasible for small $t > 0$:

1. *Affine-active constraints* (the branches that are tight): check $\langle a_j, v\rangle \ge 0$. Then they hold for all $t \ge 0$ (Lemma 7.30).

2. *Constraints not involving the moved coordinates*: here $\langle a_j, v\rangle = 0$, so there is nothing to check. This is the special case of item 1 used in Chapter 9 for the branch opposite the vertex being moved.

3. *Slack constraints* (acuteness, $D > 0$, and any branch with positive slack): they hold for all small $|t|$ by continuity, and finiteness of the list gives one threshold t_1 good for all of them (Lemma 7.29).

4. *The objective bound* $D \le D_0$: it holds automatically whenever the objective does not increase (Lemma 7.31).

5. *The objective*: $\langle g, v\rangle < 0$ gives strict decrease for small $t > 0$ (the Strict Descent Lemma 6.4 of Chapter 6).

If items 1–5 all pass, minimality is contradicted. Hence at a minimiser $\langle g, v\rangle \ge 0$ for every v passing items 1–4.

Example 7.41 (Running the checklist on Example 7.28)**.** Take the configuration of Example 7.28. The affine-active constraints are c_1 and c_2, with

$$a_1 = \big((1,0),(0,0)\big), \qquad a_2 = \big((0,-1),(0,1)\big).$$

Write $v = (v_2, v_3)$ with $v_2 = (v_2^{(1)}, v_2^{(2)})$ and $v_3 = (v_3^{(1)}, v_3^{(2)})$. Then

$$\langle a_1, v\rangle = v_2^{(1)}, \qquad \langle a_2, v\rangle = -v_2^{(2)} + v_3^{(2)},$$

so

$$V = \big\{v \in \mathbb{R}^4 :\ v_2^{(1)} \ge 0 \text{ and } v_3^{(2)} \ge v_2^{(2)}\big\}.$$

The objective gradient is $g = \big((1,-1),(0,1)\big)$, so

$$\langle g, v\rangle = v_2^{(1)} - v_2^{(2)} + v_3^{(2)} = \langle a_1, v\rangle + \langle a_2, v\rangle \ \ge\ 0 \qquad \text{for every } v \in V,$$

which verifies the conclusion of Theorem 7.36 directly, and displays the multipliers $(1, 1, 0)$ again.

Note, however, that $c_5 = \kappa_2$ is also active here and is neither affine nor an objective bound, so hypothesis (c) of Theorem 7.36 fails and the theorem does not apply as stated. The computation above shows only that the conclusion holds anyway at this point. Chapter 9 removes this obstruction by proving that the minimiser is strictly acute, so that κ_2 is slack there and Lemma 7.29 applies.

Example 7.42 (The affine descent identity for a single vertex)**.** Let $x_* = (P_2, P_3)$ be a feasible point of the problem of Example 7.26 at which no branch constraint is active, and move the single vertex P_2, holding P_3 fixed. That is, take the velocity

$$v = (-g_2,\, 0) \in \mathbb{R}^4, \qquad g_2 \;=\; \nabla_{P_2} D \;=\; -J_\sigma P_3 \;\in\; \mathbb{R}^2.$$

Here $g_2 \in \mathbb{R}^2$ is the vertex gradient of D at P_2, not the $\mathbb{R}^4$ objective gradient $g = (-J_\sigma P_3,\ J_\sigma P_2)$ of Example 7.26; item 3 below uses that only one block moves.

1. There are no affine-active constraints, so $\mathrm{Aff}(x_*) = \emptyset$, items 1 and 2 of the checklist are vacuous, and $V(x_*) = \mathbb{R}^4$; in particular $v \in V(x_*)$.

2. All branch constraints are slack, as are acuteness and $D > 0$, so by Lemma 7.29 there is $t_1 > 0$ such that all of them hold at $x_* + tv$ for $0 < t < t_1$.

3. For fixed P_3 the map $P_2 \mapsto D(P_2, P_3) = \sigma \det(P_2, P_3)$ is linear, with gradient g_2. So Lemma 6.7(iv) applies in the P_2 block and gives, for every $t \in \mathbb{R}$,

$$D(x_* + tv) = \sigma \det(P_2 - tg_2,\, P_3) = D(x_*) - t\,\sigma \det(g_2, P_3) = D(x_*) - t\,\|g_2\|^2.$$

For the last equality, $\sigma \det(u, w) = \langle J_\sigma u, w\rangle$ gives

$$\sigma \det(g_2, P_3) = \langle J_\sigma g_2, P_3\rangle = \langle -J_\sigma^2 P_3,\ P_3\rangle = \langle P_3, P_3\rangle = \|P_3\|^2 = \|g_2\|^2,$$

using $J_\sigma^2 = -I$ and the fact that J_σ is an isometry. Hence $D(x_* + tv) < D(x_*)$ for every $t > 0$: Lemma 7.31 preserves the bound $D \le D_0$, and the objective strictly decreases.

Choosing any $t \in (0, t_1)$ therefore produces a feasible point with strictly smaller objective value, contradicting minimality. Chapter 9 cites this argument for the case of a vertex with no active branch. By Corollary 6.25, $\|g_2\| = \|P_3\|$ is the length of the side of the centre triangle opposite P_2, which is at least 1, so D decreases at rate at least 1 in t.

Remark 7.43 (Why the motion must be confined to one block). The same argument with the full $\mathbb{R}^4$ gradient, $v = -g$, fails. The objective D is quadratic, and it is not affine along $-g$: expanding with $\det(J_\sigma u, J_\sigma w) = \det(u, w)$ gives

$$D(x - tg) \;=\; D(x) - t\|g\|^2 + t^2 D(x),$$

which increases for large t when $D(x) > 0$. Only the block-by-block computation above is an identity in t, and only there does Lemma 6.7(iv) apply.

Exercises

Exercise 7.1. Prove directly from Definition 7.5, without quoting the algebra of limits, that $\det : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is continuous. Show first that

$$|\det(u, v) - \det(u', v')| \;\leq\; \|u - u'\|\,\|v\| + \|u'\|\,\|v - v'\|,$$

which follows on adding and subtracting $\det(u', v)$, using bilinearity, and quoting the bound $|\det(p, q)| \leq \|p\|\,\|q\|$ of Chapter 2. Then exhibit an explicit $\delta > 0$, in terms of ε, $\|u'\|$ and $\|v'\|$, that works at the point (u', v').

Exercise 7.2. Let $\mathrm{NObt} \subseteq \mathbb{R}^4$ be the set of configurations $x = (P_2, P_3)$, with $P_1 = 0$, satisfying $\kappa_1, \kappa_2, \kappa_3 \geq 0$ — that is, non-obtuse in the sense fixed in Example 7.9 — and let $\mathrm{Acu} \subseteq \mathbb{R}^4$ be the set of those satisfying $\kappa_1, \kappa_2, \kappa_3 > 0$.

(a) Show that NObt is closed and unbounded.

(b) Show that Acu is open, and that every $x \in \mathrm{Acu}$ has its three centres pairwise distinct.

(c) Exhibit a sequence in Acu converging to a point of $\mathrm{NObt} \setminus \mathrm{Acu}$, and deduce that Acu is not closed.

(d) Exhibit a sequence in the complement of NObt converging to a point of NObt, and deduce that NObt is not open. (One such sequence is already in the text: see Example 7.13.) Explain why (c) does not by itself prove this.

(e) Show that if $x \in \mathrm{NObt}$ and $D(x) = 0$ then two of the three centres coincide. (Hint: if $\det(P_2, P_3) = 0$ then $P_2 = ae$ and $P_3 = be$ for some unit vector e and reals a, b; write the three inequalities $\kappa_i \geq 0$ in terms of a and b, and multiply two of them.) Deduce that every $x \in \mathrm{NObt}$ whose three centres are distinct has $D(x) \neq 0$.

Exercise 7.3. (a) Let (x_k) be a bounded sequence in $\mathbb{R}^m$ with the property that every convergent subsequence has the same limit L. Prove that $x_k \to L$. (Suppose not, extract a subsequence staying at distance at least ε from L, and apply Theorem 7.15 to it.)

(b) Prove the converse of Theorem 7.17: a sequentially compact subset of $\mathbb{R}^n$ is closed and bounded.

Exercise 7.4. Three limitations of the results of §§7.3–7.5.

(a) Let $C \subseteq \mathbb{R}^n$ be closed and nonempty, and put $c(x) = -\inf\{\|x - y\| : y \in C\}$. Prove that $|c(x) - c(x')| \leq \|x - x'\|$ for all x, x', hence that c is continuous, and that $C = \{x : c(x) \geq 0\}$. Deduce, treating the empty set separately, that Theorem 7.11(a) has an exact converse: the subsets of $\mathbb{R}^n$ cut out by finitely many non-strict inequalities between continuous functions are precisely the closed sets. (So the theorem, unlike Proposition 1.6, loses nothing, and correspondingly says nothing about the shape of the feasible set.)

(b) Show that the functions $c_k(x) = 1/k - x$ of Remark 7.12 converge uniformly on $\mathbb{R}$ to a continuous limit. Conclude that no uniform-convergence hypothesis rescues Theorem 7.11(b) for a countable list; finiteness is what the proof needs.

(c) Exhibit a closed unbounded $X \subseteq \mathbb{R}^2$ and a continuous $f : X \to \mathbb{R}$ that is bounded below on X but does not attain its infimum. Then prove that no such example exists if f is assumed in addition to satisfy $f(x) \to \infty$ as $\|x\| \to \infty$. (For the second part, fix any $x_0 \in X$ and apply Theorem 7.19 on $X \cap \{x : \|x\| \le R\}$ for a suitable R.)

Exercise 7.5. A second fully computed configuration, with data different from those of Example 7.28. Take $\sigma = +1$ and the frozen data

$$n_1 = \tfrac{1}{\sqrt{5}}(2,1), \quad n_2 = (0,1), \quad n_3 = (0,-1), \qquad H_1 = \tfrac{4}{\sqrt{5}}, \quad H_2 = 2, \quad H_3 = 1,$$

with $D_0 = 4$, and the configuration $P_1 = 0$, $P_2 = (2,0)$, $P_3 = (1,2)$.

(a) List all eight constraints of Example 7.26 with their numerical values here, and classify each as active or slack. Check that the centre triangle is strictly acute, so that, unlike in Example 7.28, hypothesis (c) of Theorem 7.36 is satisfied.

(b) Compute a_1, a_2, a_3 and g, and verify Euler's identity $\langle g, x\rangle = 2D$.

(c) Describe $V(x) \subseteq \mathbb{R}^4$ explicitly as an intersection of half-spaces, verify directly that $\langle g, v\rangle \ge 0$ for every $v \in V(x)$, and exhibit a nonzero $v \in V(x)$ with $\langle g, v\rangle = 0$.

(d) Find all triples $(\lambda_1, \lambda_2, \lambda_3)$ of nonnegative reals with $g = \sum_i \lambda_i a_i$; show there is exactly one, and check that $\sum_i \lambda_i H_i = 2D$ for it. Which of the three multipliers is forced to vanish, and why could that have been predicted before computing anything?

(As in Example 7.28, the data are chosen for arithmetic convenience.)

Exercise 7.6. Consider on $\mathbb{R}^2$ the single constraint $c(y) = -y_1^2 \ge 0$ and the objective $f(y) = -y_1$.

(a) Determine the feasible set X, and show that every point of X is a global minimiser of f over X.

(b) At $x_* = 0$, compute $\nabla c(0)$ and the set $\{v : \langle \nabla c(0), v\rangle \ge 0\}$, and exhibit a vector v in that set with $\langle \nabla f(0), v\rangle < 0$.

(c) Identify which hypothesis of Theorem 7.36 fails at $x_* = 0$, and explain why this configuration is therefore not a counterexample to the theorem. Observe also that the conclusion of the theorem, read with $\mathrm{Aff}(x_*) = \emptyset$ and hence $V(x_*) = \mathbb{R}^2$, is false here, so that this hypothesis cannot be dropped.

Exercise 7.7. Let x_* be a feasible point of $X = \{y : c_j(y) \ge 0,\ j = 1, \dots, N\}$, where f is differentiable at x_*, every c_j is continuous at x_*, and every constraint active at x_* is affine.

(a) Suppose in addition that no constraint is active at x_* and that f is affine with gradient $g \ne 0$. Prove that x_* is not a minimiser of f over X, by exhibiting an explicit $t > 0$ for which $x_* - tg$ is feasible with strictly smaller objective value. (The t_1 supplied by Lemma 7.29 and the identity $f(x_* - tg) = f(x_*) - t\|g\|^2$ may be used.)

(b) Suppose instead that x_* is a minimiser and that exactly one constraint is active at x_*, with gradient $a \ne 0$; write $g = \nabla f(x_*)$. Prove that $\langle g, v\rangle \ge 0$ for every v with $\langle a, v\rangle \ge 0$, and deduce, without any theory of cones, that $g = \mu a$ for some $\mu \ge 0$. (Hint: apply the conclusion to v and to $-v$ for every v orthogonal to a, then apply it to $v = a$.)

Chapter 8

Cones, Separation, and the Farkas Lemma

What this chapter assumes

- From Chapter 2: the quarter turn J_σ and the two identities $J_\sigma^2 = -I$ and $\langle J_\sigma u, v\rangle = \sigma \det(u, v)$; in the plane, the orthogonal complement of a unit vector n is the line spanned by $J_\sigma n$.
- From Chapter 4: convex sets, convex combinations, half-planes. Cones are defined below: sets containing 0 and closed under scaling by $t \geq 0$; those used here are in addition convex.
- From Chapter 6: gradients and directional derivatives, and in particular that $\langle g, v\rangle < 0$ forces f to decrease strictly along the ray $t \mapsto x_0 + tv$ for all small $t > 0$; the distinction between linear and affine; the block inner product on $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$; the branch functions $b_1(x) = \langle P_2, n_1\rangle$, $b_2(x) = \langle P_3 - P_2, n_2\rangle$, $b_3(x) = \langle -P_3, n_3\rangle$ and their constant gradients $a_1 = (n_1, 0)$, $a_2 = (-n_2, n_2)$, $a_3 = (0, -n_3)$.
- From Chapter 7: a set cut out by finitely many non-strict inequalities between continuous functions is closed, and one cut out by strict inequalities is open. Bolzano–Weierstrass in $\mathbb{R}^m$ (Theorem 7.15), the sequential compactness of a closed bounded set (Theorem 7.17), and the extreme value theorem (Theorem 7.19). Active versus slack constraints. The first-order optimality theorem, Theorem 7.36: at a minimiser x_* one has $\langle g, v\rangle \geq 0$ for every admissible velocity v. Also the two ingredients of its proof, Lemma 7.30(i) and Lemma 7.31.

Notation for this chapter

The ambient Euclidean space is written $\mathbb{R}^N$, since n, n_1, n_2, n_3 denote unit normals. From Section 8.4 onwards, C denotes a nonempty closed convex set, and β the separation constant of Theorem 8.20; these are unrelated to the thresholds and certificate data of Chapter 12. Here μ is the half-plane multiplier of Lemma 8.12 and c the number $\min_i\langle a_i, w\rangle$ of Section 8.6. Cones are written $\Gamma = \operatorname{cone}(a_1, \dots, a_m)$, and λ_i denotes a multiplier or, more generally, a coefficient in a conic combination.

Chapter 7 ended with Theorem 7.36: at a minimiser there is no feasible descent direction, an infinite family of inequalities $\langle g, v\rangle \geq 0$, one for each admissible velocity v. This chapter replaces that family by a finite algebraic statement, $g = \sum_i \lambda_i a_i$ with all $\lambda_i \geq 0$; the replacement is a duality theorem.

8.1 Convex cones

Definition 8.1. A set $\Gamma \subseteq \mathbb{R}^N$ is a *cone* if $0 \in \Gamma$ and $tx \in \Gamma$ whenever $x \in \Gamma$ and $t \geq 0$. It is a *convex cone* if it is in addition convex.

Lemma 8.2. *A nonempty set $\Gamma \subseteq \mathbb{R}^N$ is a convex cone if and only if it is closed under nonnegative scaling and under addition.*

Proof. Suppose Γ is closed under both operations. Taking $t = 0$ shows $0 \in \Gamma$ (the set is nonempty), so Γ is a cone; and for $x, y \in \Gamma$ and $t \in [0,1]$ the vectors tx and $(1-t)y$ lie in Γ, hence so does their sum $tx + (1-t)y$, so Γ is convex. Conversely, let Γ be a convex cone. Closure under nonnegative scaling is part of the definition. If $x, y \in \Gamma$ then $\frac{1}{2}(x+y) \in \Gamma$ by convexity, and $x + y = 2 \cdot \frac{1}{2}(x+y) \in \Gamma$ by scaling. □

Definition 8.3. A *conic combination* of $a_1, \dots, a_m \in \mathbb{R}^N$ is a vector $\sum_{i=1}^m \lambda_i a_i$ with $\lambda_1, \dots, \lambda_m \geq 0$. We write $\operatorname{cone}(a_1, \dots, a_m)$ for the set of all such combinations.

Proposition 8.4. *$\Gamma = \operatorname{cone}(a_1, \dots, a_m)$ is a convex cone containing each a_i, and it is contained in every convex cone containing $a_1, \dots, a_m$.*

Proof. If $x = \sum_i \lambda_i a_i$ and $y = \sum_i \lambda_i' a_i$ are conic combinations and $t \geq 0$, then $tx = \sum_i (t\lambda_i) a_i$ and $x + y = \sum_i (\lambda_i + \lambda_i') a_i$ are conic combinations, since the coefficients $t\lambda_i$ and $\lambda_i + \lambda_i'$ are nonnegative. By Lemma 8.2, Γ is a convex cone. It contains a_i, the combination with $\lambda_i = 1$ and all other coefficients 0. Finally, let Γ' be a convex cone containing every a_j. Then $\lambda_i a_i \in \Gamma'$ for each i by scaling, and adding these m vectors one at a time keeps us inside Γ', so $\sum_i \lambda_i a_i \in \Gamma'$. Hence $\Gamma \subseteq \Gamma'$. □

Remark 8.5 (Cones and subspaces). A subspace is closed under negation; a cone is not. For $a \neq 0$, $\operatorname{cone}(a)$ is a ray and $\operatorname{span}(a)$ is a line. The velocity sets produced by inequality constraints are wedges rather than subspaces. Testing a linear functional against a wedge yields a one-sided conclusion, a nonnegative multiplier; testing it against a subspace yields an equation with a multiplier of unrestricted sign. This is the origin of the sign conditions in Chapters 10–13.

Example 8.6 (The first quadrant)**.** Let $a_1 = (1,0)$ and $a_2 = (0,1)$ in $\mathbb{R}^2$. Then

$$\Gamma = \operatorname{cone}(a_1, a_2) = \{(x_1, x_2) : x_1 \geq 0,\ x_2 \geq 0\},$$

the closed first quadrant: the conic combination $\lambda_1 a_1 + \lambda_2 a_2$ is (λ_1, λ_2), and every point of the quadrant arises once this way. Two observations recur below. First, $-a_1 = (-1, 0) \notin \Gamma$, so the cone is not a subspace. Second, the set of v satisfying $\langle a_1, v\rangle \geq 0$ and $\langle a_2, v\rangle \geq 0$ is again the closed first quadrant, so here the cone and the set of directions it tests nonnegatively coincide. This example is the case $m = 2$ of the Farkas lemma below.

Example 8.7 (A cone that is a subspace)**.** Let $a_1 = (1,0)$ and $a_2 = (-1,0)$ in $\mathbb{R}^2$. Then $\operatorname{cone}(a_1, a_2) = \{(x_1, 0) : x_1 \in \mathbb{R}\}$ is the whole x_1-axis, a subspace. Note for later use that no vector w satisfies both $\langle a_1, w\rangle > 0$ and $\langle a_2, w\rangle > 0$, since $\langle a_1, w\rangle = -\langle a_2, w\rangle$; the hypothesis introduced in Section 8.6 fails here.

Example 8.8 (The branch cone)**.** Let n_1, n_2, n_3 be unit vectors in $\mathbb{R}^2$ and set, in $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$ with the block inner product,

$$a_1 = (n_1, 0), \qquad a_2 = (-n_2, n_2), \qquad a_3 = (0, -n_3).$$

For $x = (P_2, P_3) \in \mathbb{R}^4$ we compute directly from the block inner product

$$\langle a_1, x\rangle = \langle n_1, P_2\rangle, \qquad \langle a_3, x\rangle = \langle n_3, -P_3\rangle,$$

$$\langle a_2, x\rangle = -\langle n_2, P_2\rangle + \langle n_2, P_3\rangle = \langle n_2, P_3 - P_2\rangle,$$

that is,

$$\langle a_i, x\rangle = b_i(x) \qquad (i = 1,2,3) \tag{8.1}$$

for the branch functions of Chapter 6. The identity uses that the b_i are linear rather than affine: for an affine $b(x) = \langle a, x\rangle + k$ with $k \neq 0$, the left side of (8.1) would omit the constant k. Chapter 10 obtains a strict dual vector from this identity by taking $w = x$.

The generators a_1, a_2, a_3 are linearly independent as soon as n_1 and n_2 are non-parallel; this is the block computation of Example 8.30 below.

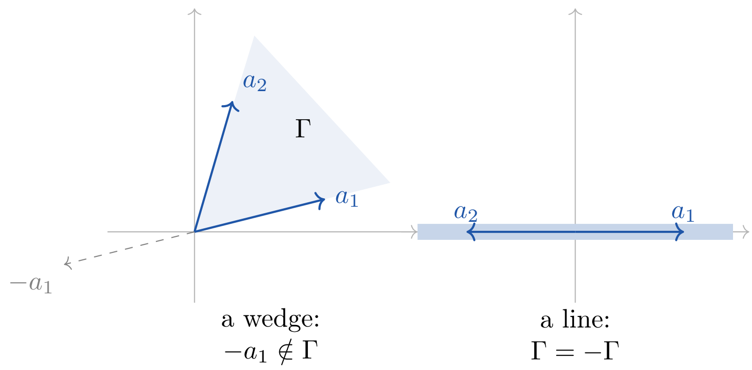


Figure 8.1: Two finitely generated cones in $\mathbb{R}^2$. Left: a wedge $\Gamma = \mathrm{cone}(a_1, a_2)$ with independent generators; the reflected generator $-a_1$ lies outside it. Right: the degenerate case $a_2 = -a_1$ of Example 8.7, where the cone is a line.

8.2 The cone of admissible velocities

Definition 8.9. Given $a_1, \ldots, a_m \in \mathbb{R}^N$, the *feasible-direction cone* is

$$\Gamma^* = \{v \in \mathbb{R}^N : \ \langle a_i, v\rangle \geq 0 \text{ for } i = 1, \ldots, m\}.$$

Proposition 8.10. *Γ^* is a closed convex cone.*

Proof. If $\langle a_i, v\rangle \geq 0$ and $t \geq 0$ then $\langle a_i, tv\rangle = t\langle a_i, v\rangle \geq 0$, and if also $\langle a_i, v'\rangle \geq 0$ then $\langle a_i, v+v'\rangle = \langle a_i, v\rangle + \langle a_i, v'\rangle \geq 0$; since $0 \in \Gamma^*$, Lemma 8.2 makes Γ^* a convex cone. It is cut out by the m non-strict inequalities $\langle a_i, \cdot\rangle \geq 0$ between continuous functions, hence closed by Chapter 7. □

Remark 8.11. Γ^* is called the *dual cone* of $\Gamma = \mathrm{cone}(a_1, \ldots, a_m)$; the Farkas lemma of Section 8.7 says that, under one extra hypothesis, $(\Gamma^*)^* = \Gamma$. Only the name is used; no further theory of duality is needed.

With the constraints $b_i(x) = \langle a_i, x\rangle \geq H_i$ all active at a feasible point x_*, the set Γ^* is the set $V(x_*)$ of admissible velocities of Definition 7.34, and Theorem 7.36 says: if x_* minimises f and f is differentiable at x_* with $g = \nabla f(x_*)$, then

$$\langle g, v\rangle \geq 0 \qquad \text{for every } v \in \Gamma^*. \tag{8.2}$$

Two ingredients of that theorem's proof are used again below. For an active linear constraint, Lemma 7.30(i) gives feasibility along the whole ray $x_* + tv$, $t \geq 0$, not only for small t; this fails for active non-linear constraints (Remark 7.32), which is why Chapter 6 normalises $P_1 = 0$ to make the branch functions linear. A bound on the objective itself obstructs only ascent steps (Lemma 7.31). Condition (8.2) is hypothesis (ii) of the Farkas lemma, Theorem 8.27.

8.3 Half-plane duality

The general case is treated in Section 8.7. The case $m = 1$ admits a short direct proof; it is the form used in Chapter 9, and settles part (b) of Exercise 7.7 of Chapter 7.

Lemma 8.12 (Half-plane duality)**.** *Let $n \in \mathbb{R}^N$ be a unit vector and $g \in \mathbb{R}^N$. If $\langle g, v\rangle \geq 0$ for every v with $\langle n, v\rangle \geq 0$, then $g = \mu n$ with $\mu = \langle g, n\rangle \geq 0$.*

Proof. Suppose $\langle n, v\rangle = 0$. Then $\langle n, -v\rangle = 0 \geq 0$ as well, so both v and $-v$ satisfy the hypothesis, giving $\langle g, v\rangle \geq 0$ and $-\langle g, v\rangle \geq 0$, whence $\langle g, v\rangle = 0$. Thus g is orthogonal to $n^\perp$. Write the orthogonal decomposition $g = \mu n + h$ with $\mu = \langle g, n\rangle$ and $h = g - \mu n \in n^\perp$ (indeed $\langle h, n\rangle = \langle g, n\rangle - \mu\|n\|^2 = 0$). Applying the previous sentence to $v = h$ gives $0 = \langle g, h\rangle = \mu\langle n, h\rangle + \|h\|^2 = \|h\|^2$, so $h = 0$ and $g = \mu n$. Finally $v = n$ satisfies $\langle n, n\rangle = 1 \geq 0$, so $\mu = \langle g, n\rangle \geq 0$. □

Remark 8.13 (The planar reading). In $\mathbb{R}^2$ the subspace $n^\perp$ is the line spanned by $J_\sigma n$ (Chapter 2), so the test vectors used in the first step of the proof are $\pm J_\sigma n$, and the lemma reads: if g has nonnegative inner product with the closed half-plane $\langle n, v\rangle \geq 0$, then g lies on the ray $\mathbb{R}_{\geq 0} n$.

Remark 8.14 (The case $m = 1$ of the Farkas lemma). Here $\Gamma = \mathrm{cone}(n)$ is a ray, Γ^* is the closed half-space $\langle n, v\rangle \geq 0$, and the conclusion says that $g \in \Gamma$. The proof used neither a closedness argument (a ray is closed) nor a separation theorem. The orthogonal decomposition it relies on has no analogue for m generators; that is what Sections 8.4–8.6 supply.

Corollary 8.15. *Under the hypotheses of Lemma 8.12, if in addition $\|g\| \geq 1$ then $\mu \geq 1$.*

Proof. $\|g\| = |\mu|\,\|n\| = |\mu| = \mu$, since $\mu \geq 0$ and $\|n\| = 1$. □

In Chapter 9 this corollary is applied at a vertex of the centre triangle with one active branch: there $\|g\| \geq 1$ is the area-gradient bound of Chapter 6, and $\mu \geq 1$ combines with the threshold bound $H \geq 1$ to give $D = \mu H \geq 1$.

Example 8.16 (Testing the hypothesis)**.** Take $g = (1, 2)$ and $n = (0, 1)$ in $\mathbb{R}^2$. The vector $v = (1, 0)$ satisfies $\langle n, v\rangle = 0 \geq 0$, and so does $-v$; but $\langle g, v\rangle = 1$ while $\langle g, -v\rangle = -1 < 0$. So the hypothesis of Lemma 8.12 fails, as it must, since g is not a multiple of n. In optimisation terms, $-v$ is a feasible descent direction: by the contrapositive of the lemma, if g does not lie on the normal ray then some admissible direction decreases f.

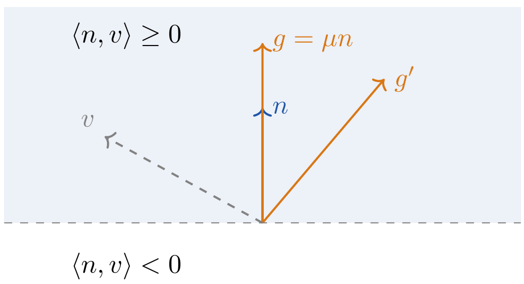


Figure 8.2: Half-plane duality. The admissible half-plane is shaded, its boundary dashed. An admissible gradient g lies on the blue normal ray. The gradient g' does not, and the grey direction v certifies failure: $\langle n, v\rangle \geq 0$ but $\langle g', v\rangle < 0$, a feasible descent direction.

8.4 Nearest points in a closed convex set

Lemma 8.17 (Parallelogram law)**.** *For $u, w \in \mathbb{R}^N$,*

$$\|u\|^2 + \|w\|^2 = 2\left\|\frac{u+w}{2}\right\|^2 + \frac{1}{2}\|u - w\|^2.$$

Proof. By bilinearity of the inner product, $\|u + w\|^2 = \|u\|^2 + 2\langle u, w\rangle + \|w\|^2$ and $\|u - w\|^2 = \|u\|^2 - 2\langle u, w\rangle + \|w\|^2$. Adding and halving gives $\frac{1}{2}\|u + w\|^2 + \frac{1}{2}\|u - w\|^2 = \|u\|^2 + \|w\|^2$, and $\frac{1}{2}\|u + w\|^2 = 2\|(u + w)/2\|^2$. □

Theorem 8.18 (Nearest-point projection)**.** *Let $C \subseteq \mathbb{R}^N$ be nonempty, closed and convex, and let $g \in \mathbb{R}^N$. Then there is exactly one $p \in C$ with $\|p - g\| \leq \|x - g\|$ for all $x \in C$. Writing $\mathrm{dist}(g, C) = \inf_{x \in C}\|x - g\|$ for the distance from g to C, one has $\mathrm{dist}(g, C) = \|p - g\|$.*

Proof. Existence. Fix any $x_0 \in C$ and put $R = \|x_0 - g\|$. Let

$$C' = C \cap \{x : \|x - g\|^2 \leq R^2\},$$

which contains x_0 and so is nonempty. It is closed, being the intersection of the closed set C with a set cut out by one non-strict inequality between continuous functions (Chapter 7), and it is bounded, since $\|x\| \le \|g\| + R$ on C'. Being nonempty, closed and bounded, C' is sequentially compact by Theorem 7.17, and the continuous function $x \mapsto \|x - g\|$ attains a minimum on it, at some $p \in C'$, by the extreme value theorem, Theorem 7.19. If $x \in C \setminus C'$ then $\|x - g\| > R = \|x_0 - g\| \ge \|p - g\|$, so p minimises over all of C.

Uniqueness. Let $p, q \in C$ both attain the minimum value $\delta = \|p - g\| = \|q - g\|$. By convexity $\frac{1}{2}(p+q) \in C$, so $\|\frac{1}{2}(p+q) - g\| \ge \delta$. Apply Lemma 8.17 with $u = p - g$ and $w = q - g$, noting $(u+w)/2 = \frac{1}{2}(p+q) - g$ and $u - w = p - q$:

$$2\delta^2 = \|u\|^2 + \|w\|^2 = 2\left\|\tfrac{1}{2}(p+q) - g\right\|^2 + \tfrac{1}{2}\|p - q\|^2 \ge 2\delta^2 + \tfrac{1}{2}\|p - q\|^2.$$

Hence $\|p - q\|^2 \le 0$, so $p = q$. □

Existence used only that C is nonempty and that the slice C' is closed and bounded; uniqueness used only convexity.

Proposition 8.19 (Variational characterisation)**.** *Let $C \subseteq \mathbb{R}^N$ be convex, $g \in \mathbb{R}^N$ and $p \in C$. Then*

$$\begin{aligned} & \|p - g\| \le \|x - g\| \quad \textit{for every } x \in C \\ \iff \quad & \langle g - p,\ x - p\rangle \le 0 \quad \textit{for every } x \in C. \end{aligned}$$

Proof. ($\Rightarrow$) Let $x \in C$ and $t \in (0, 1]$. By convexity $p + t(x - p) \in C$, so

$$\|p - g\|^2 \le \|p - g + t(x - p)\|^2 = \|p - g\|^2 + 2t\langle p - g,\ x - p\rangle + t^2\|x - p\|^2.$$

Cancel $\|p - g\|^2$ and divide by $t > 0$ to get $2\langle p - g, x - p\rangle + t\|x - p\|^2 \ge 0$. Letting $t \to 0^+$ gives $\langle p - g, x - p\rangle \ge 0$, i.e. $\langle g - p, x - p\rangle \le 0$.

($\Leftarrow$) For $x \in C$,

$$\|x - g\|^2 = \|(x - p) + (p - g)\|^2 = \|x - p\|^2 + 2\langle x - p,\ p - g\rangle + \|p - g\|^2 \ge \|p - g\|^2,$$

since $\langle x - p, p - g\rangle = -\langle g - p, x - p\rangle \ge 0$. □

When C is closed the point p exists and is unique (Theorem 8.18), and the second inequality characterises it; for merely convex C no minimiser need exist. Geometrically, every direction from p into C makes a non-acute angle with $g - p$.

8.5 Separating a point from a closed convex set

Theorem 8.20 (Point/closed-convex separation)**.** *Let $C \subseteq \mathbb{R}^N$ be nonempty, closed and convex, and let $g \notin C$. Then there exist $v \in \mathbb{R}^N$ and $\beta \in \mathbb{R}$ with*

$$\langle x, v\rangle \ge \beta \quad \textit{for all } x \in C, \qquad \textit{and} \qquad \langle g, v\rangle < \beta.$$

One may take $v = p - g$, where p is the nearest point of C to g, and $\beta = \langle p, v\rangle$; then $\beta - \langle g, v\rangle = \|v\|^2 = \mathrm{dist}(g, C)^2 > 0$.

Proof. Let p be the nearest point of C to g, which exists and is unique by Theorem 8.18, and set $v = p - g$ and $\beta = \langle p, v\rangle$. Since $g \notin C$ and $p \in C$ we have $v \ne 0$, so $\|v\|^2 > 0$. Proposition 8.19 gives $\langle g - p, x - p\rangle \le 0$ for all $x \in C$, that is $\langle v, x - p\rangle \ge 0$, that is

$$\langle x, v\rangle \ge \langle p, v\rangle = \beta \qquad (x \in C).$$

On the other hand

$$\langle g, v\rangle = \langle p, v\rangle - \langle p - g, v\rangle = \beta - \|v\|^2 < \beta.$$

Finally $\|v\| = \|p - g\| = \mathrm{dist}(g, C)$ by the choice of p. □

Remark 8.21 (Geometry of the separating hyperplane). The hyperplane $\{y : \langle y, v\rangle = \beta - \frac{1}{2}\|v\|^2\}$ is perpendicular to the segment from g to p and contains its midpoint $\frac{1}{2}(p+g)$, since $\langle \frac{1}{2}(p+g), v\rangle = \frac{1}{2}(\beta + \beta - \|v\|^2)$. The set C lies strictly on one side of it and g strictly on the other.

Example 8.22 (Convexity is necessary)**.** Let $C = \{x \in \mathbb{R}^2 : \|x\| = 1\}$, the unit circle, which is closed but not convex, and let $g = 0 \notin C$. Suppose (v, β) satisfied the conclusion. From $\langle g, v\rangle = 0 < \beta$ we get $\beta > 0$. If $v = 0$ then $\langle x, v\rangle = 0 < \beta$ for $x \in C$, a contradiction. If $v \neq 0$ then $x = -v/\|v\| \in C$ and $\langle x, v\rangle = -\|v\| < 0 < \beta$, again a contradiction. So no separating pair exists. Closedness is likewise necessary; see Exercise 8.6.

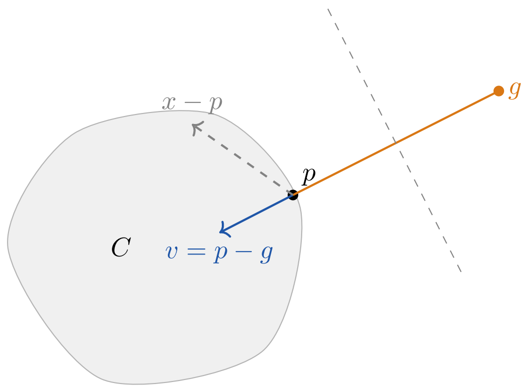


Figure 8.3: Separating a point from a closed convex set. The nearest point p realises $\mathrm{dist}(g, C)$; the separating line (dashed) is perpendicular to the segment pg and crosses it at its midpoint. The vector $v = p - g$ points from g towards C, the sign convention of the theorem: $\langle x, v\rangle \geq \beta > \langle g, v\rangle$ for $x \in C$.

8.6 Closedness of a finitely generated cone given a strict dual vector

Definition 8.23. A vector $w \in \mathbb{R}^N$ is a *strict dual vector* for $a_1, \ldots, a_m$ if $\langle a_i, w\rangle > 0$ for every $i = 1, \ldots, m$.

Theorem 8.24. *Suppose that $a_1, \ldots, a_m$ admit a strict dual vector w. Then the cone $\Gamma = \mathrm{cone}(a_1, \ldots, a_m)$ is closed.*

Proof. Put $c = \min_{1\leq i\leq m}\langle a_i, w\rangle$, a minimum of finitely many strictly positive numbers, so $c > 0$. Let $x_k \in \Gamma$ with $x_k \to x$, and write $x_k = \sum_{i=1}^m \lambda_{i,k} a_i$ with all $\lambda_{i,k} \geq 0$.

The coefficients are bounded. Pairing with w,

$$\langle x_k, w\rangle = \sum_{i=1}^m \lambda_{i,k}\langle a_i, w\rangle \;\geq\; c\sum_{i=1}^m \lambda_{i,k},$$

using $\lambda_{i,k} \geq 0$ and $\langle a_i, w\rangle \geq c$. A convergent sequence is bounded, say $\|x_k\| \leq M_0$ for all k, so by Cauchy–Schwarz $\langle x_k, w\rangle \leq M_0\|w\| =: M$. Hence $\sum_i \lambda_{i,k} \leq M/c$ for every k, and in particular each coefficient vector $\Lambda_k = (\lambda_{1,k}, \ldots, \lambda_{m,k})$ lies in the bounded subset $\{\Lambda \in \mathbb{R}^m : \Lambda_i \geq 0,\ \sum_i \Lambda_i \leq M/c\}$ of $\mathbb{R}^m$.

Passing to a limit. By Bolzano–Weierstrass in $\mathbb{R}^m$ (Chapter 7) some subsequence converges, $\Lambda_{k_j} \to \Lambda = (\lambda_1, \ldots, \lambda_m)$. Each $\lambda_i \geq 0$, since non-strict inequalities are preserved by limits. The map $\Lambda \mapsto \sum_i \Lambda_i a_i$ is linear, hence continuous, so

$$\sum_{i=1}^m \lambda_i a_i = \lim_{j\to\infty}\sum_{i=1}^m \lambda_{i,k_j} a_i = \lim_{j\to\infty} x_{k_j} = x.$$

Therefore $x \in \Gamma$, and Γ is closed. $\square$

Remark 8.25 (Role of the strict dual vector). The strict dual vector was used only to bound the coefficients; without it they can be unbounded. With $a_1 = (1,0)$ and $a_2 = (-1,0)$ as in Example 8.7, the constant sequence $x_k = k\,a_1 + k\,a_2 = 0$ has coefficient sum $2k \to \infty$, and no strict dual vector exists.

In that example the cone is closed anyway. Every finitely generated cone in $\mathbb{R}^N$ is closed: a cone is finitely generated if and only if it is the solution set of finitely many homogeneous linear inequalities. This is the conic form of the Minkowski–Weyl theorem, quoted without proof in Chapter 4, Remark 4.7, and not used here. Chapter 10 obtains the hypothesis of Theorem 8.24 by taking $w = x$, the dilation direction: by (8.1), $\langle a_i, x\rangle = b_i(x) = H_i \geq 1 > 0$ at complete contact.

Lemma 8.26 (Pointedness)**.** *Let w be a strict dual vector for $a_1, \ldots, a_m$ and put $\Gamma = \operatorname{cone}(a_1, \ldots, a_m)$. Then:*

(i) $\langle x, w\rangle > 0$ for every nonzero $x \in \Gamma$; hence $\Gamma \cap (-\Gamma) = \{0\}$, and Γ contains no line through the origin;

(ii) Γ contains no line at all.

Proof. (i) Let $x = \sum_i \lambda_i a_i \in \Gamma$ with all $\lambda_i \geq 0$. Then $\langle x, w\rangle = \sum_i \lambda_i \langle a_i, w\rangle \geq 0$, with equality only if $\lambda_i\langle a_i, w\rangle = 0$ for every i, i.e. only if every $\lambda_i = 0$, i.e. only if $x = 0$. If now $x \in \Gamma \cap (-\Gamma)$ then $\langle x, w\rangle \geq 0$ and $\langle -x, w\rangle \geq 0$, so $\langle x, w\rangle = 0$ and hence $x = 0$. A line through the origin is $\{tz : t \in \mathbb{R}\}$ with $z \neq 0$, and if it lay in Γ then z and $-z$ would both lie in $\Gamma \cap (-\Gamma)$.

(ii) Suppose Γ contained a line $L = \{q + tz : t \in \mathbb{R}\}$ with $z \neq 0$, not necessarily through the origin. For $t > 0$ both $q + tz$ and $q - tz$ lie in Γ, hence so do $\frac{1}{t}(q + tz) = \frac{1}{t}q + z$ and $\frac{1}{t}q - z$. By Theorem 8.24 the cone Γ is closed, so letting $t \to \infty$ gives $z \in \Gamma$ and $-z \in \Gamma$, contradicting $\Gamma \cap (-\Gamma) = \{0\}$. ☐

Part (ii) is stronger than the last clause of (i), and its proof uses the closedness of Γ.

8.7 The finite Farkas lemma

Theorem 8.27 (Finite Farkas lemma)**.** *Let $a_1, \ldots, a_m$ and g be vectors in $\mathbb{R}^N$. Suppose*

(i) there is a vector w with $\langle a_i, w\rangle > 0$ for every i;

(ii) every v with $\langle a_i, v\rangle \geq 0$ for all i satisfies $\langle g, v\rangle \geq 0$.

Then $g = \sum_{i=1}^m \lambda_i a_i$ for some $\lambda_1, \ldots, \lambda_m \geq 0$; that is, $g \in \operatorname{cone}(a_1, \ldots, a_m)$.

Proof. Write $\Gamma = \operatorname{cone}(a_1, \ldots, a_m)$.

Step 1: Γ is closed and convex. Convexity is Proposition 8.4; closedness follows from hypothesis (i) and Theorem 8.24. Also Γ is nonempty, since $0 \in \Gamma$.

Step 2: separation. Suppose for contradiction that $g \notin \Gamma$. By Theorem 8.20 there are v and β with $\langle x, v\rangle \geq \beta$ for all $x \in \Gamma$ and $\langle g, v\rangle < \beta$.

Step 3: $\langle\cdot, v\rangle$ is nonnegative on Γ. Since $0 \in \Gamma$, taking $x = 0$ gives $\beta \leq 0$. Moreover $\langle x, v\rangle \geq 0$ for every $x \in \Gamma$: if some $x_0 \in \Gamma$ had $\langle x_0, v\rangle < 0$, then $tx_0 \in \Gamma$ for all $t > 0$ and $\langle tx_0, v\rangle = t\langle x_0, v\rangle \to -\infty$ as $t \to \infty$, contradicting the lower bound β.

Step 4: contradiction with (ii). Each a_i lies in Γ, so Step 3 gives $\langle a_i, v\rangle \geq 0$ for every i. Hypothesis (ii) then forces $\langle g, v\rangle \geq 0$. But $\langle g, v\rangle < \beta \leq 0$, a contradiction. Hence $g \in \Gamma$. ☐

Step 3 is the only step that uses that Γ is a cone rather than merely convex; it is the source of the one-sided conclusion.

Proposition 8.28 (The converse, and the alternative)**.** *If $g = \sum_{i=1}^m \lambda_i a_i$ with all $\lambda_i \geq 0$, then*

$$\langle g, v\rangle = \sum_{i=1}^{m} \lambda_i \langle a_i, v\rangle \geq 0$$

for every v with $\langle a_i, v\rangle \geq 0$ for all i. Consequently, assuming (i) of Theorem 8.27, exactly one of the following holds:

(a) $g \in \operatorname{cone}(a_1, \dots, a_m)$*;*

(b) there is v *with* $\langle a_i, v\rangle \geq 0$ *for all* i *and* $\langle g, v\rangle < 0$*.*

Proof. The displayed computation follows from bilinearity and the sign of each term. It shows that (a) implies the negation of (b); and Theorem 8.27 says that the negation of (b), which is hypothesis (ii), implies (a). So (a) and (b) are mutually exclusive and, under (i), exhaustive. □

Statements of this shape are called *theorems of the alternative*: either the desired representation exists, or there is a witness to its impossibility. The multipliers in (a) are a finite *certificate* that no feasible descent direction exists. Chapter 12 uses certificates of a different kind.

The active cone

With constraints $b_i(x) \geq H_i$, feasibility restricts x to directions v satisfying $\langle a_i, v\rangle \geq 0$ for the active indices i; the slack ones impose no local restriction, by Chapter 7. The Farkas lemma then says the objective gradient g lies in the wedge spanned by the active constraint gradients. If g fell outside that wedge, some admissible direction v would have $\langle g, v\rangle < 0$ and the point would not be minimal.

Example 8.29 (The quadrant, continued)**.** Take $a_1 = (1, 0)$, $a_2 = (0, 1)$ as in Example 8.6, so $\Gamma = \Gamma^*$ is the closed first quadrant, and $w = (1, 1)$ is a strict dual vector, since $\langle a_1, w\rangle = \langle a_2, w\rangle = 1 > 0$. For $g = (2, 3)$ the multipliers are $\lambda_1 = 2$, $\lambda_2 = 3$. For $g = (2, -1)$ we have $g \notin \Gamma$, and the alternative is witnessed by $v = (0, 1)$: indeed $\langle a_1, v\rangle = 0 \geq 0$ and $\langle a_2, v\rangle = 1 \geq 0$, while $\langle g, v\rangle = -1 < 0$. Read back as optimisation, this is the problem "minimise f subject to $x_1 \geq 0$ and $x_2 \geq 0$" at the origin, where both constraints are active: minimality forces both components of $g = \nabla f(0)$ to be nonnegative, and if one is negative, sliding along the corresponding axis decreases f without leaving the quadrant.

Example 8.30 (Uniqueness of the multipliers)**.** Take $a_1 = (1, 0)$, $a_2 = (0, 1)$, $a_3 = (1, 1)$ and $g = (1, 1)$ in $\mathbb{R}^2$. Both $(\lambda_1, \lambda_2, \lambda_3) = (1, 1, 0)$ and $(0, 0, 1)$ are valid nonnegative multiplier vectors. The Farkas lemma asserts existence only.

If $a_1, \dots, a_m$ are linearly independent the multipliers are unique, since two representations of g differ by a vanishing linear combination of independent vectors. Apply this to the generators of Example 8.8. Suppose n_1 and n_2 are not parallel, and that $t_1 a_1 + t_2 a_2 + t_3 a_3 = 0$ in $\mathbb{R}^4$. Reading the two blocks separately,

$$t_1 n_1 - t_2 n_2 = 0, \qquad t_2 n_2 - t_3 n_3 = 0.$$

If $t_1 \neq 0$ then $n_1 = (t_2/t_1) n_2$, making n_1, n_2 parallel; so $t_1 = 0$, whence $t_2 n_2 = 0$ and $t_2 = 0$ (as $\|n_2\| = 1$), whence $t_3 n_3 = 0$ and $t_3 = 0$. Thus a_1, a_2, a_3 are linearly independent as soon as $n_1 \nparallel n_2$, and the multipliers produced in Chapter 10 are unique in the case treated in Chapters 11 and 12, where all the oriented sines s_i are positive and hence no two normals are parallel.

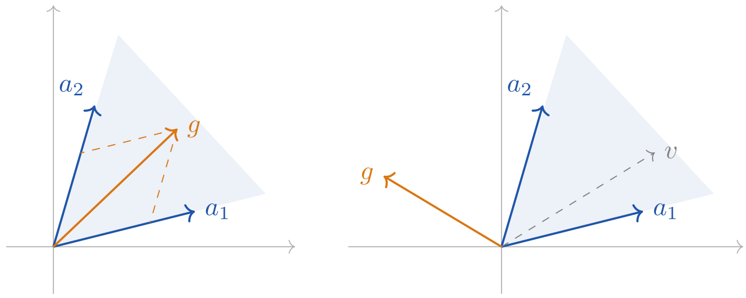
8.8 Farkas, Lagrange, and the sign of the multipliers

Lemma 8.31 (The subspace analogue)**.** *Let* $V \subseteq \mathbb{R}^N$ *be a linear subspace and* $g \in \mathbb{R}^N$*. If* $\langle g, v\rangle \geq 0$ *for all* $v \in V$*, then* $\langle g, v\rangle = 0$ *for all* $v \in V$*, i.e.* $g \in V^\perp$*.*

Proof. For $v \in V$ we also have $-v \in V$, so $\langle g, v\rangle \geq 0$ and $-\langle g, v\rangle \geq 0$. □

Equality constraints give unsigned multipliers. If a constraint reads $b(x) = H$ with $b(x) = \langle a, x\rangle$ linear, then the velocities preserving it form the subspace $V = \{v : \langle a, v\rangle = 0\}$, and Lemma 8.31 gives $g \in V^\perp = \operatorname{span}(a)$, i.e. $g = \lambda a$ with $\lambda \in \mathbb{R}$ of either sign. This is the Lagrange multiplier rule for a single linear equality constraint.

Inequality constraints give signed multipliers. If the constraint reads $b(x) \geq H$, the admissible velocities form a half-space, a wedge, and the test $v \mapsto -v$ is no longer available. Half-plane duality (Lemma 8.12) then yields $g = \mu a$ with $\mu \geq 0$, and the Farkas lemma is the m-constraint version of the same phenomenon. The signs depend on how the constraints are oriented, so we fix a convention:



$g = \lambda_1 a_1 + \lambda_2 a_2,\ \lambda_i \geq 0$ $\qquad$ $\langle a_i, v\rangle \geq 0$ but $\langle g, v\rangle < 0$

Figure 8.4: The Farkas alternative. Left: g lies in the shaded cone $\operatorname{cone}(a_1, a_2)$ and the nonnegative multipliers are read off by resolving g along the generators. Right: g lies outside, and $v \in \Gamma^*$ certifies this by $\langle g, v\rangle < 0$.

> write every constraint as $b_i(x) \geq H_i$, with b_i increasing in the feasible direction, and minimise; then g lies in the cone generated by the gradients ∇b_i, with nonnegative coefficients.

With "maximise" in place of "minimise", or $\leq$ in place of $\geq$, the signs in the conclusion reverse.

Active constraints only. Chapter 7 showed that a slack constraint imposes no local restriction, so it contributes no generator to the cone; equivalently one may include it as a generator and give it multiplier 0. This is *complementary slackness.* In our application it is vacuous: Chapter 9 proves complete contact, so all three frozen branches are active at the minimiser and all three generators are present.

Theorem 8.27 is due to Farkas (1902). The general first-order rule for inequality-constrained minimisation is the stationarity condition of the Karush–Kuhn–Tucker theorem (Karush 1939; Kuhn and Tucker 1951), in which hypothesis (i) plays the role of a *constraint qualification.* Only the linear-constraint case proved above is used here.

In Chapter 10 the lemma is applied with $m = 3$ in $\mathbb{R}^4$. Hypothesis (i) comes from $w = x$ via Example 8.8. Hypothesis (ii) is checked there constraint by constraint, by the argument of Chapter 7, §7.8: the three branch constraints are active and linear, the acuteness constraints and $D > 0$ are slack and hence open, and the objective bound $D \leq D_0$ is preserved by any descent step. The resulting multipliers $\lambda_1, \lambda_2, \lambda_3 \geq 0$ become the forces $f_i = \lambda_i n_i$.

Exercises

Exercise 8.1. (a) Let $a_1, \dots, a_m$ be nonzero and let $\operatorname{conv}(a_1, \dots, a_m)$ denote their convex hull. Show that

$$\operatorname{cone}(a_1, \dots, a_m) = \{0\} \cup \{ty : \ t > 0,\ y \in \operatorname{conv}(a_1, \dots, a_m)\},$$

so that a finitely generated cone is the union of the rays through the points of a convex hull. (b) Show that the union of two rays $\mathbb{R}_{\geq 0}a$ and $\mathbb{R}_{\geq 0}b$ emanating from the origin in $\mathbb{R}^2$ (with $a, b \neq 0$) is always a cone, and determine exactly when it is a convex cone. (c) Show that $\Gamma = \operatorname{cone}(a_1, \dots, a_m)$ is a linear subspace if and only if $-a_i \in \Gamma$ for every i.

Exercise 8.2. Let $a_1, \dots, a_m \in \mathbb{R}^N$ and let Γ^* be as in Definition 8.9. (a) Show that every strict dual vector w for $a_1, \dots, a_m$ lies in $\operatorname{int}\Gamma^*$; deduce that if a strict dual vector exists then $\operatorname{int}\Gamma^* \neq \emptyset$. [Note first that $a_i \neq 0$ for every i, and set $c = \min_i \langle a_i, w\rangle > 0$.] (b) Conversely, suppose no a_i is 0 and $\operatorname{int}\Gamma^* \neq \emptyset$. Show that every $w \in \operatorname{int}\Gamma^*$ is a strict dual vector. [Hint: if $B(w, \rho) \subseteq \Gamma^*$, apply the defining inequality of Γ^* to the point $w - \frac{\rho}{2} a_i/\|a_i\|$.] Thus, for generators none of which vanish, the strict dual vectors are exactly the interior points of Γ^*.

Exercise 8.3. (a) Let n_1, n_2 be non-parallel unit vectors in $\mathbb{R}^2$ and suppose $\langle g, v\rangle \geq 0$ whenever $\langle n_1, v\rangle \geq 0$ and $\langle n_2, v\rangle \geq 0$. Prove directly, without using Theorem 8.27, that $g = \lambda_1 n_1 + \lambda_2 n_2$ with $\lambda_1, \lambda_2 \geq 0$. [Hint: n_1, n_2 form a basis of $\mathbb{R}^2$; let v_1 satisfy $\langle n_1, v_1\rangle = 0$ and $\langle n_2, v_1\rangle = 1$, and let v_2

satisfy $\langle n_1, v_2\rangle = 1$ and $\langle n_2, v_2\rangle = 0$. Both are admissible.] (b) Let $n \in \mathbb{R}^2$ be a unit vector and $g \in \mathbb{R}^2$ with $\|g\| \geq 1$ and $\langle g, v\rangle \geq 0$ whenever $\langle n, v\rangle \geq 0$. Suppose $r \in \mathbb{R}^2$ satisfies $\langle n, r\rangle = H \geq 1$ and $\langle g, r\rangle = D$. Show $D \geq 1$.

Exercise 8.4. (a) Find the nearest point of the closed first quadrant in $\mathbb{R}^2$ to $g = (-1, 3)$, and exhibit an explicit pair (v, β) as in Theorem 8.20. (b) Let $C = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$ and $g = (3, 4)$. Show that the nearest point of C to g is $g/\|g\|$, and compute v and β explicitly. (c) In (b), verify directly that $\beta - \langle g, v\rangle = \mathrm{dist}(g, C)^2$.

Exercise 8.5. Let $a_1 = (1, 1)$ and $a_2 = (1, -1)$ in $\mathbb{R}^2$. (a) Show that $\mathrm{cone}(a_1, a_2)$ is the set of x with $x_1 \geq |x_2|$, and find a strict dual vector. (b) Determine Γ^*. (c) For $g = (3, 1)$, find the multipliers. (d) For $g = (1, 3)$, show that $g \notin \mathrm{cone}(a_1, a_2)$ and exhibit v with $\langle a_i, v\rangle \geq 0$ for $i = 1, 2$ and $\langle g, v\rangle < 0$.

Exercise 8.6. (a) Let $C = \{x \in \mathbb{R}^2 : x_1 > 0\}$ and $g = (0, 0)$. Show that C is convex and $g \notin C$, but that no pair (v, β) satisfies the conclusion of Theorem 8.20; identify which hypothesis fails. (b) With $a_1 = (1, 0)$ and $a_2 = (-1, 0)$ in $\mathbb{R}^2$, show that no strict dual vector exists, exhibit a sequence $x_k \in \mathrm{cone}(a_1, a_2)$ converging to a point whose coefficient vectors are unbounded, and verify nevertheless that $\mathrm{cone}(a_1, a_2)$ is closed. Conclude that hypothesis (i) of Theorem 8.27 is needed for the proof given of Theorem 8.24, not for its conclusion.

Exercise 8.7. Let n_1, n_2, n_3 be unit vectors in $\mathbb{R}^2$, no two parallel, and set $a_1 = (n_1, 0)$, $a_2 = (-n_2, n_2)$, $a_3 = (0, -n_3)$ in $\mathbb{R}^4$ with the block inner product. (a) Show that a_1, a_2, a_3 are linearly independent. (b) Let $x = (P_2, P_3)$ satisfy $\langle n_1, P_2\rangle = H_1$, $\langle n_2, P_3 - P_2\rangle = H_2$ and $\langle n_3, -P_3\rangle = H_3$ with all $H_i \geq 1$. Show that $w = x$ is a strict dual vector for a_1, a_2, a_3. (c) With x as in (b), suppose $g \in \mathbb{R}^4$ satisfies $\langle g, v\rangle \geq 0$ for every v with $\langle a_i, v\rangle \geq 0$, $i = 1, 2, 3$. Writing $g = (g^{(2)}, g^{(3)})$ in block form, show that there are $\lambda_1, \lambda_2, \lambda_3 \geq 0$ with $g^{(2)} = \lambda_1 n_1 - \lambda_2 n_2$ and $g^{(3)} = \lambda_2 n_2 - \lambda_3 n_3$. [Use (b) to check hypothesis (i) of Theorem 8.27; part (a) alone does not supply it.] (d) Deduce, for the x of (b), that $\langle g, x\rangle = \lambda_1 H_1 + \lambda_2 H_2 + \lambda_3 H_3$.

Exercise 8.8 (Carathéodory for cones)**.** Let $x \in \mathrm{cone}(a_1, \ldots, a_m)$. Show that x is a conic combination of a linearly independent subfamily of $a_1, \ldots, a_m$. [Hint: take a representation $x = \sum_i \lambda_i a_i$ using as few generators as possible. If the generators used are linearly dependent, there are θ_i, not all zero, with $\sum_i \theta_i a_i = 0$; after replacing θ by $-\theta$ if necessary, some $\theta_i > 0$. Subtract t times this relation from the representation, where $t = \min\{\lambda_i/\theta_i : \theta_i > 0\}$, and check that all coefficients remain nonnegative while at least one becomes zero.] Deduce that $\mathrm{cone}(a_1, \ldots, a_m)$ is a finite union of cones generated by linearly independent subfamilies.

Chapter 9

The Freeze: The Extremal Configuration and Complete Contact

This chapter begins the proof of Theorem 1.1, which is by contradiction: a counterexample is assumed, and a rigid configuration is extracted from it. The freeze fixes such a counterexample, selects one owned separating branch for each of the three pairs of squares, and declares the frames, normals, thresholds, owners and the orientation sign constant, leaving only the three centres free. The doubled area is then minimised over the resulting feasible set, which is nonempty and compact; a minimiser exists, has strictly acute centre triangle, and satisfies $0 < D \le D_0 < 1$. Finally, complete contact: at that minimiser all three frozen inequalities are equalities. Chapter 10 starts from there.

What this chapter assumes

- **From Chapter 1.** The unit square $Q(F, P)$ with frame F and centre P, and an admissible family, that is, one with pairwise disjoint interiors. Non-obtuseness of a triple of pairwise distinct points, and the equivalence of its three forms: angle at most $\pi/2$, the inner-product form $\langle P_j - P_i,\, P_k - P_i\rangle \ge 0$, and the squared-side form, together with the corresponding equivalence for strict acuteness (Proposition 1.25). Three pairwise distinct collinear points are not non-obtuse (Lemma 1.28). The centres of two unit squares with disjoint interiors are at distance at least 1 (centre-separation lemma 1.32).

- **From Chapter 2.** The orientation sign $\sigma \in \{-1, +1\}$ and the quarter turn J_σ, of which only $J_\sigma^2 = -I$ and $\langle J_\sigma u, v\rangle = \sigma \det(u, v)$ are used. The Cauchy–Schwarz companion $\|d\| \ge \langle d, n\rangle$ for a unit vector n. The subcritical triangle lemma: a non-obtuse triangle with all sides at least 1 and doubled area less than 1 has all sides less than $\sqrt{2}$ and is strictly acute.

- **From Chapters 4 and 5.** The projected half-width $w_F(n)$, its evenness $w_F(-n) = w_F(n)$, the value $w_F(n) = \frac{1}{2}$ when n is an axis of F, and the universal bound $w_F(n) \ge \frac{1}{2}$. The oriented separating branch $\langle Q - P, n\rangle \ge H(F, G; n) = w_F(n) + w_G(n)$, which is sufficient for disjointness of the interiors of $Q(F, P)$ and $Q(G, Q)$. The owned separating branch lemma 5.14: two unit squares with disjoint interiors admit a branch whose unit normal is, up to sign, an axis of one of the two frames, its *owner*. The obstacle theorem 4.29: the interiors meet if and only if $Q - P \in \operatorname{int} Z$ for the zonogon $Z = Q(F, 0) + Q(G, 0)$. The octagon theorem 4.39, describing $\operatorname{int} Z$ by the eight strict inequalities $\langle x, n\rangle < w_F(n) + w_G(n)$, one for each axis n of F or of G, and the identification $h_Z(n) = w_F(n) + w_G(n)$ of the support function (Corollary 4.33). The frame circle $\mathbb{R}/(\pi/2)\mathbb{Z}$ with frame distance $d_4(F, G) \in [0, \pi/4]$, and the owned-branch threshold formula $H(F, G; n) = H\big(d_4(F, G)\big)$ with $H(t) = (1 + \cos t + \sin t)/2 \ge 1$.

- **From Chapter 6.** Translation invariance and the normalisation $P_1 = 0$, with the configuration point $x = (P_2, P_3) \in \mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$ and its block inner product. The vertex dictionary (Proposition 6.12), which translates a motion of one centre into a velocity of x. The branch functions

b_1, b_2, b_3, linear with constant gradients $a_1 = (n_1, 0)$, $a_2 = (-n_2, n_2)$, $a_3 = (0, -n_3)$. The affine difference identity $f(x) - f(y) = \langle g, x - y\rangle$ and its consequence $f(x - tg) = f(x) - t\|g\|^2$. The vertex-gradient theorem 6.22: with the other two centres held fixed, D is an affine function of P_j with constant gradient $\nabla_{P_j} D = J_\sigma d_{j+1}$, the quarter turn of the opposite edge.

- **From Chapter 7.** A set cut out by finitely many non-strict inequalities between continuous functions is closed, and one cut out by strict inequalities is open, so strict conditions persist under small perturbations. Bolzano–Weierstrass in $\mathbb{R}^m$, the extreme value theorem, and the vocabulary of active and slack constraints. The eight-constraint list of the minimisation problem, written out as Example 7.26. The three perturbation lemmas 7.29 (slack constraints impose no local restriction), 7.30 (an active affine constraint confines velocities to a half-space and then survives along the whole ray) and 7.31 (an objective bound never obstructs descent). The first-order optimality theorem 7.36, that a minimiser admits no feasible descent direction, with its Corollary 7.38 for a single active constraint. Remark 7.33, that moving one centre is a legitimate motion of x despite the normalisation.

- **From Chapter 8.** The half-plane lemma: if n is a unit vector and $\langle g, v\rangle \geq 0$ for every v with $\langle n, v\rangle \geq 0$, then $g = \mu n$ with $\mu = \langle g, n\rangle \geq 0$; and its corollary that $\|g\| \geq 1$ forces $\mu \geq 1$. The Farkas lemma is not used in this chapter.

9.1 The assumed counterexample

Everything in Chapters 9–12 is proved under one hypothesis, which Chapter 13 discharges by deriving a contradiction from it.

> **Standing Hypothesis (C).** There exist unit squares $S_i = Q(F_i, P_i)$, $i = 1, 2, 3$, with pairwise disjoint interiors, whose centre triangle $P_1P_2P_3$ is non-obtuse and satisfies
>
> $$2\,\mathrm{Area}(P_1P_2P_3) \;<\; 1.$$

Hypothesis (C) is the negation of Theorem 1.1 for one triple.

Lemma 9.1 (The counterexample is nondegenerate)**.** *Assume (C). Then P_1, P_2, P_3 are pairwise distinct and noncollinear. Consequently* $\det(P_2 - P_1, P_3 - P_1) \neq 0$*, the sign*

$$\sigma \;=\; \operatorname{sign}\det(P_2 - P_1,\, P_3 - P_1) \;\in\; \{-1, +1\}$$

is well defined, and the number $D_0 := \sigma\det(P_2 - P_1, P_3 - P_1)$ *satisfies* $0 < D_0 < 1$.

Proof. Any two of the squares have disjoint interiors, so by the centre-separation lemma 1.32 of Chapter 1 their centres are at distance at least 1; in particular the three centres are pairwise distinct. They are therefore not collinear: three pairwise distinct collinear points are not non-obtuse (Lemma 1.28 of Chapter 1), whereas (C) assumes the centre triangle non-obtuse.

Noncollinearity means that $P_2 - P_1$ and $P_3 - P_1$ are linearly independent, so their determinant is nonzero and σ is defined. Then $D_0 = \sigma\det(P_2 - P_1, P_3 - P_1) = |\det(P_2 - P_1, P_3 - P_1)| = 2\,\mathrm{Area}(P_1P_2P_3)$, which is positive, and less than 1 by (C). ☐

Definition 9.2 (Frozen orientation)**.** The sign σ of Lemma 9.1 is computed at the configuration of (C) and held fixed thereafter. For an arbitrary triple of points P_1, P_2, P_3 we write

$$D \;=\; \sigma\det(P_2 - P_1,\, P_3 - P_1)$$

and call it the *doubled signed area*, or simply the doubled area. It equals $2\,\mathrm{Area}(P_1P_2P_3)$ exactly when it is nonnegative, and $-2\,\mathrm{Area}(P_1P_2P_3)$ otherwise.

Remark 9.3 (Why the sign is frozen). The signed form $\sigma\det(P_2 - P_1, P_3 - P_1)$ has three properties that $|\det(P_2 - P_1, P_3 - P_1)|$ lacks. First, D is a polynomial in the coordinates, so the two constraints $c_7 = D \geq 0$ and $c_8 = D_0 - D \geq 0$ of the frozen problem are polynomial, hence continuous, and the feasible set is closed in Theorem 9.16. Second, D has constant gradient $\nabla_{P_j} D = J_\sigma d_{j+1}$ (Chapter 6), used in every descent argument below. Third, D is homogeneous of degree 2 in x, the hypothesis of Euler's identity, which Chapter 10 uses to convert multipliers into area.

This requires one extra constraint, $D \geq 0$; Lemma 9.15 below shows that it is strict on the feasible set, so it obstructs nothing. On the feasible set $|\det| = D$, so the two objectives agree there; they differ in that only the signed one is polynomial, differentiable and homogeneous on all of $\mathbb{R}^4$.

9.2 The freeze

Definition 9.4 (The freeze)**.** Assume (C), and fix σ and D_0 as in Lemma 9.1. The *frozen data* are constructed as follows.

(1) *Select a branch for each pair.* For each of the three pairs (S_1, S_2), (S_2, S_3), (S_3, S_1), the squares have disjoint interiors, so the owned separating branch lemma of Chapter 5 supplies a unit vector which is, up to sign, an axis of one of the two frames of the pair, and for which the branch inequality holds. Choose one such vector for each pair. If several exist (Example 9.7), choose one arbitrarily.

(2) *Orient it cyclically.* Replace the chosen vector for the pair (S_i, S_{i+1}) by its negative if necessary, so that the branch inequality is written off the cyclic edge $d_i = P_{i+1} - P_i$. Call the result n_i. This is legitimate because w_F is even in its argument (Chapter 5), so $H(F, G; -n) = H(F, G; n)$ and the threshold is unchanged; and $\langle Q - P, n\rangle \geq H$ becomes $\langle P - Q, -n\rangle \geq H$.

(3) *Record the threshold.* Set

$$H_i \;:=\; H(F_i, F_{i+1}; n_i) \;=\; w_{F_i}(n_i) + w_{F_{i+1}}(n_i).$$

By the threshold formula of Chapter 5, $H_i = H\big(d_4(F_i, F_{i+1})\big) \geq 1$, and this value does not depend on which of the two frames supplied the axis.

(4) *Fix an owner.* Choose, for each i, a square of the pair whose frame has n_i as an axis, and call it the *owner* of branch i. If both squares qualify, choose one arbitrarily and keep it.

The frames F_1, F_2, F_3, the unit normals n_1, n_2, n_3, the thresholds H_1, H_2, H_3, the owners, the sign σ and the number D_0 are now *frozen*: they are constants for the remainder of the book. Only the centres vary, subject to the *frozen system*

$$\langle d_1, n_1\rangle \;\geq\; H_1, \qquad \langle d_2, n_2\rangle \;\geq\; H_2, \qquad \langle d_3, n_3\rangle \;\geq\; H_3, \tag{9.1}$$

where $d_1 = P_2 - P_1$, $d_2 = P_3 - P_2$, $d_3 = P_1 - P_3$, so that $d_1 + d_2 + d_3 = 0$.

Remark 9.5 (What is constant and what varies). The symbols divide as follows.

Frozen constants	**Variables**
frames F_1, F_2, F_3	centres P_1, P_2, P_3
unit normals n_1, n_2, n_3	equivalently $x = (P_2, P_3)$ after $P_1 = 0$
thresholds $H_1, H_2, H_3 \geq 1$	
the owner of each of the three branches	
the orientation sign σ	
the number $D_0 \in (0, 1)$	

No quantity in the left column is a function of anything in the right column. The thresholds H_i are determined by the frames alone, and retain their meaning as the centres move.

Remark 9.6 (The normals are not functions of the edges). Each n_i is an axis of a frozen frame. It need not be parallel to d_i, need not be perpendicular to d_i, and does not move when the centres move. The inequality $\langle d_i, n_i\rangle \geq H_i$ is the only relation between n_i and d_i that is used. The normals in Figure 9.1 are drawn askew to the edges because n_i need not be aligned with d_i, nor with the outward normal of the triangle.

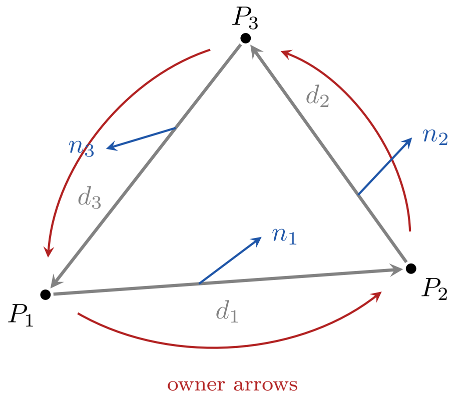


Figure 9.1: The frozen system (9.1): the cyclic edges d_1, d_2, d_3 (grey), the frozen branch normals n_1, n_2, n_3 (blue, drawn askew to the edges because they are axes of frames fixed in advance, not functions of the edges), and the ownership arrows (red), each pointing from the owner of a branch to the other endpoint and bent away from the edges so as not to be confused with d_i. The ownership shown, branch i owned by S_i, is one possibility among several.

Example 9.7 (An explicit freeze)**.** Let all three frames be the standard frame $F = (e_1, e_2)$, and take the axis-parallel unit squares centred at

$$P_1 = (0,0), \qquad P_2 = (1,0), \qquad P_3 = (0,1).$$

These are the three squares $[-\frac{1}{2}, \frac{1}{2}]^2$, $[\frac{1}{2}, \frac{3}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$ and $[-\frac{1}{2}, \frac{1}{2}] \times [\frac{1}{2}, \frac{3}{2}]$; the first two meet only along the segment $x = \frac{1}{2}$, and similarly for the other pairs, so the interiors are pairwise disjoint. The triangle has a right angle at P_1 and is therefore non-obtuse, and

$$\sigma = \operatorname{sign}\det\big((1,0),(0,1)\big) = +1, \qquad D = \sigma\det\big((1,0),(0,1)\big) = 1.$$

Every frame here is the standard frame, so $d_4(F_i, F_j) = 0$ and every owned threshold equals $H(0) = (1 + \cos 0 + \sin 0)/2 = 1$; equivalently, $w_F(n) = \frac{1}{2}$ for every axis n, so $H = \frac{1}{2} + \frac{1}{2} = 1$.

Now carry out steps (1)–(2) of Definition 9.4. The cyclic edges are $d_1 = (1,0)$, $d_2 = (-1,1)$, $d_3 = (0,-1)$, and the candidate normals are the four axes $\pm e_1, \pm e_2$.

- Pair (S_1, S_2): $\langle d_1, (1,0)\rangle = 1 \geq 1$, so $n_1 = (1,0)$ is a valid oriented branch. The other three axes give $\langle d_1, (-1,0)\rangle = -1$, $\langle d_1, (0,\pm1)\rangle = 0$, all < 1. The choice is forced.
- Pair (S_2, S_3): $\langle d_2, (-1,0)\rangle = 1 \geq 1$ and $\langle d_2, (0,1)\rangle = 1 \geq 1$. Two valid choices, $n_2 = (-1,0)$ or $n_2 = (0,1)$.
- Pair (S_3, S_1): $\langle d_3, (0,-1)\rangle = 1 \geq 1$, and no other axis works. The choice is forced.

Take $n_1 = (1,0)$, $n_2 = (-1,0)$, $n_3 = (0,-1)$, all with $H_i = 1$; every square owns every branch here, since all frames coincide, so step (4) is an arbitrary choice too.

The owned branch need not be unique, so step (1) of the freeze fixes a choice; the later chapters compute with the branch selected rather than with a canonical one. And at this configuration all three branch inequalities hold with equality and $D = 1$, so the bound is sharp: the contradiction requires the strict inequality $D < 1$ and cannot be drawn from $D \leq 1$.

Remark 9.8 (The examples do not satisfy (C))*.* Theorem 1.1 is true, so no configuration satisfying (C) exists; the freeze is carried out inside a proof by contradiction. Every explicit example in this chapter, including Example 9.7 and the computational Example 9.27, therefore has $D \geq 1$ and illustrates the construction rather than the hypothesis. What such an example can check are the identities: gradients, branch values, activity, the equation $D = \mu H$. These hold for any data of the frozen shape, whether or not it arises from a counterexample, and that is how they are used.

9.3 The relaxation; contact versus activity

Lemma 9.9 (Feasible triples are admissible)**.** *Let F_1, F_2, F_3, n_1, n_2, n_3, H_1, H_2, H_3 be the frozen data of Definition 9.4, and let P_1, P_2, P_3 be any three points satisfying the frozen system* (9.1)*.*

Then the three unit squares $Q(F_i, P_i)$, $i = 1, 2, 3$, have pairwise disjoint interiors.

Proof. Fix i and consider the pair $Q(F_i, P_i)$, $Q(F_{i+1}, P_{i+1})$. The i-th inequality of (9.1) reads

$$\langle P_{i+1} - P_i,\ n_i \rangle \ \geq\ H_i \ =\ w_{F_i}(n_i) + w_{F_{i+1}}(n_i),$$

which is the oriented separating branch of Chapter 5 for this pair, with unit normal n_i. By the sufficiency half of the branch lemma, the two squares have disjoint interiors.

The threshold H_i depends only on the frames, which are frozen, so it remains the threshold for the pair at any position of the centres. □

Remark 9.10 (The frozen system and disjointness). The frozen system is stronger than disjointness, not weaker: a branch with a given normal is sufficient for disjoint interiors but not necessary. Some branch always exists for a disjoint pair, by the owned separating branch lemma of Chapter 5, but it need not be the one we froze, and once the centres have moved the frozen normal may certify nothing. The set of centre triples satisfying (9.1) is therefore contained in the set of triples for which the squares $Q(F_i, P_i)$ are admissible, generally properly. What the freeze relaxes is the difficulty of the problem, in three ways.

- With the frames fixed, disjointness of the interiors of $Q(F, P)$ and $Q(G, Q)$ says $Q - P \notin \operatorname{int} Z$, and by the octagon theorem of Chapter 4 that is a disjunction of the eight linear inequalities $\langle Q - P, n\rangle \geq w_F(n) + w_G(n)$, one for each axis n of F or of G, that is, one for each of the eight candidate branches. (There are four rather than eight when the two frames share their axes.) A disjunction of half-planes is not convex, and it is not a system of simultaneous inequalities. The freeze commits to one disjunct per pair, replacing three disjunctions of eight by three simultaneous linear inequalities on a convex set.
- The frames stop being variables. The rotational degrees of freedom, three of them, disappear, and the configuration becomes a point of $\mathbb{R}^4$ rather than a point of a space of triples of convex bodies.
- The objective becomes a polynomial, with constant-gradient structure in each vertex separately.

The argument needs only: (i) that the original configuration is still feasible, so that the feasible set is nonempty; (ii) that the feasible set is compact; and (iii) that the strict geometric conditions used in the descent arguments hold at the minimiser. Lemma 9.9 is not required for any of this, and after the freeze squares do not reappear in the argument. Its role is interpretive: it shows that the minimiser is an actual configuration of three unit squares with pairwise disjoint interiors.

Remark 9.11 (Active does not mean tangent).

A frozen branch is *active* at a configuration when the corresponding inequality in (9.1) holds with equality. This is a statement about numbers, not about squares. It does not assert that the two squares touch. Squares are never moved or rotated: only the centres vary, and the frames, normals, thresholds, owners and σ are constants.

Example 9.12 (An active branch between distant squares)**.** Let $F = G$ be the standard frame and take $P = (0, 0)$, $Q = (1, 5)$, with normal $n = (1, 0)$. Then $H = w_F(n) + w_G(n) = \frac{1}{2} + \frac{1}{2} = 1$ and $\langle Q - P, n\rangle = 1 = H$: the branch is active. But the squares are $[-\frac{1}{2}, \frac{1}{2}]^2$ and $[\frac{1}{2}, \frac{3}{2}] \times [\frac{9}{2}, \frac{11}{2}]$, whose boundaries are at distance 4. Activity of a branch says that the shadows of the two squares on the n-axis touch; it says nothing about the other coordinate.

The converse implication does hold: if two unit squares with disjoint interiors have boundaries that meet, then every valid branch for that pair is active. That is Exercise 9.5. So tangency of squares implies activity of every branch, but not conversely.

9.4 Proof by minimal counterexample

Minimal counterexamples. To prove a universally quantified statement one may assume a counterexample, select among all counterexamples an extremal one, use extremality to force structure, and contradict something with that structure. Over $\mathbb{N}$ the selection step is immediate: every nonempty set of naturals has a least element. Over a continuum there need be no least element, so the selection must be arranged. The tool is the extreme value theorem of Chapter 7: a continuous function on a nonempty closed bounded subset of $\mathbb{R}^m$ attains its minimum. The construction of this chapter (normalisation by translation to a point of $\mathbb{R}^4$, the replacement of disjointness by three linear inequalities, and the cap on the objective) makes that theorem applicable.

One could instead minimise directly over all admissible triples of unit squares with the frames free to rotate. That space is not in any obvious way a subset of an $\mathbb{R}^m$, and the disjunction of eight linear inequalities described in Remark 9.10 then varies, in its disjuncts and in their number, with the frames. Rather than develop a theory of convergence for such families, we freeze the frames and pick one disjunct per pair, reducing the problem to four real variables and eight simultaneous polynomial inequalities.

The two uses of hypothesis (C). Hypothesis (C) is used twice. First, it provides a feasible point, so the feasible set is nonempty; without it the minimisation is over the empty set and says nothing. Second, it provides the number $D_0 < 1$, and the constraint $D \le D_0$ bounds the feasible set. Without that constraint the frozen system has feasible points of arbitrarily large area (Exercise 9.3), so the feasible set is unbounded and no minimiser need exist.

- The proof cannot be run directly. One cannot minimise D over all configurations with the given frozen data and show the minimum is at least 1; there may be no minimum. It must be run by contradiction, with the counterexample supplying both the point and the cap.
- Capping at $D \le 1$ instead of $D \le D_0 < 1$ would fail, and one can say where. Step 4 of the proof of Theorem 9.26 derives $D = \mu H \ge 1$ at a vertex of active degree 1; under the cap $D \le 1$ this is consistent, so no contradiction is available and the degree-1 case cannot be excluded. That the constant cannot be improved is shown by Example 9.7, where a frozen system is satisfied with $D = 1$.

Outline of the proof.

1. (Chapter 9) Assume (C); fix σ and D_0; freeze one owned branch per pair.

2. (Chapter 9) Minimise D over the frozen feasible set: a minimiser exists, its centre triangle is strictly acute, and $0 < D \le D_0 < 1$ there.

3. (Chapter 9) At the minimiser all three frozen branches are active: *complete contact.*

4. (Chapter 10) The Farkas lemma supplies $\lambda_i \ge 0$ with $g = \sum_i \lambda_i a_i$; Euler homogeneity converts multipliers into area, $2D = \lambda_1 H_1 + \lambda_2 H_2 + \lambda_3 H_3$; and resolving the multiplier equation into planar components and dotting with the normals gives the sine system. Complete contact puts the constants H_i on the left of these equations.

5. (Chapters 10–12) Three exhaustive cases each give $\lambda_1 H_1 + \lambda_2 H_2 + \lambda_3 H_3 \ge 2$, hence $D \ge 1$.

6. (Chapter 13) $D \ge 1$ and $D \le D_0 < 1$ contradict each other.

Two points about the logic. (a) The final inequality $D \ge 1$ is a statement about the minimiser, not about the configuration assumed in (C). The contradiction comes from bounding the same quantity D, the doubled area at the minimiser, in two ways, and step 2 is what ties the minimiser back to D_0. (b) The case analysis of step 5 must be exhaustive; this is checked in Chapter 10 and rechecked in Chapter 13.

9.5 The frozen minimisation problem

Lemma 9.13 (Translation invariance)**.** *Let $\delta \in \mathbb{R}^2$ and replace (P_1, P_2, P_3) by $(P_1+\delta, P_2+\delta, P_3+\delta)$. Then every cyclic edge d_i, every branch value $\langle d_i, n_i\rangle$, every inner product κ_j of the two edge vectors at a vertex, and the doubled area D are unchanged.*

Proof. Each listed quantity is a function of the differences $P_j - P_k$ alone, and these are unchanged. This is the translation invariance lemma of Chapter 6. □

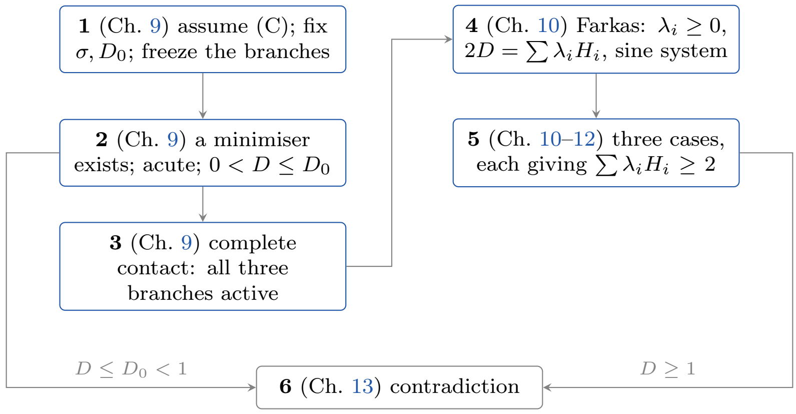


Figure 9.2: The structure of the proof. The two arrows into the final node carry two bounds on the same number, the doubled area at the minimiser.

By Lemma 9.13 a configuration may be recorded by the normalised point

$$x = (P_2 - P_1,\ P_3 - P_1) \in \mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2,$$

and we write $P_1 = 0, P_2, P_3$ for its representative, with the block inner product of Chapter 6 on $\mathbb{R}^4$. The normalisation does not forbid moving P_1, on which Section 9.8 depends. Moving P_1 by h and renormalising is the velocity $v = (-h, -h)$, while moving P_2 or P_3 by h is $v = (h, 0)$ or $v = (0, h)$; this is the vertex dictionary, Proposition 6.12 of Chapter 6, and Remark 7.33 of Chapter 7 records that the renormalisation changes neither feasibility nor the objective.

Definition 9.14 (The frozen problem)**.** Let the frozen data be as in Definition 9.4. For $x = (P_2, P_3) \in \mathbb{R}^4$, with $P_1 = 0$, define

- the *objective* $D(x) = \sigma \det(P_2, P_3)$, a homogeneous quadratic polynomial;
- the *branch functions*

  $$b_1(x) = \langle P_2, n_1\rangle, \qquad b_2(x) = \langle P_3 - P_2, n_2\rangle, \qquad b_3(x) = \langle -P_3, n_3\rangle,$$

  each linear with constant gradient $a_1 = (n_1, 0)$, $a_2 = (-n_2, n_2)$, $a_3 = (0, -n_3)$, and satisfying $b_i(x) = \langle d_i, n_i\rangle$ (Chapter 6);
- the *non-obtuseness functions* $\kappa_1, \kappa_2, \kappa_3$, where κ_j is the inner product of the two edge vectors emanating from P_j:

  $$\kappa_1(x) = \langle P_2, P_3\rangle, \qquad \kappa_2(x) = \langle -P_2,\ P_3 - P_2\rangle = \|P_2\|^2 - \langle P_2, P_3\rangle,$$
  $$\kappa_3(x) = \langle -P_3,\ P_2 - P_3\rangle = \|P_3\|^2 - \langle P_2, P_3\rangle.$$

  (The edge vectors at $P_1 = 0$ are P_2 and P_3; at P_2 they are $P_1 - P_2 = -P_2$ and $P_3 - P_2$; at P_3 they are $-P_3$ and $P_2 - P_3$.) For pairwise distinct centres, the centre triangle is non-obtuse if and only if $\kappa_1, \kappa_2, \kappa_3 \ge 0$, and strictly acute if and only if $\kappa_1, \kappa_2, \kappa_3 > 0$: this is the equivalence of the angle form with the inner-product form, Proposition 1.25 of Chapter 1, applied at each vertex. The distinctness proviso matters. The definition of X below imposes the three inequalities $\kappa_j \ge 0$ as they stand, and these are meaningful at degenerate points: at $P_2 = P_3 = 0$ all three vanish. The geometric reading applies only once Lemma 9.15(ii) has produced pairwise distinct centres at every feasible point.

The *feasible set* is

$$X \;=\; \{x \in \mathbb{R}^4 :\ b_i(x) \ge H_i\ (i = 1, 2, 3),\quad \kappa_j(x) \ge 0\ (j = 1, 2, 3),\quad 0 \le D(x) \le D_0\},$$

and the *frozen problem* is to minimise D over X.

This is the problem of Example 7.26 of Chapter 7, whose constraint numbering we adopt: in "≥ 0" form the eight constraints are $c_1 = b_1 - H_1$, $c_2 = b_2 - H_2$, $c_3 = b_3 - H_3$, $c_4 = \kappa_1$, $c_5 = \kappa_2$, $c_6 = \kappa_3$, $c_7 = D$ and $c_8 = D_0 - D$. All are non-strict and polynomial in the four coordinates of x, and only the first three are affine.

9.6 Existence and geometry of the minimum

Lemma 9.15 (Every feasible triple is a subcritical triangle)**.** *Let $x \in X$. Then:*

(i) $\|d_i\| \ \geq\ \langle d_i, n_i\rangle\ \geq\ H_i\ \geq\ 1$ *for* $i = 1, 2, 3$*;*

(ii) the three centres are distinct and noncollinear; hence $D(x) \neq 0$*, so* $0 < D(x) \leq D_0 < 1$*, and* $D(x)$ *is the doubled area* $2\,\mathrm{Area}(P_1P_2P_3)$*;*

(iii) every side has length less than $\sqrt{2}$*; in particular* $\|P_2\| < \sqrt{2}$ *and* $\|P_3\| < \sqrt{2}$*;*

(iv) the centre triangle is strictly acute: $\kappa_j(x) > 0$ *for* $j = 1, 2, 3$*.*

Proof. (i) The first inequality is the Cauchy–Schwarz companion $\|d\| \geq \langle d, n\rangle$ of Chapter 2, valid because n_i is a unit vector; the second is the branch constraint $b_i(x) \geq H_i$; the third is the universal threshold bound $H_i \geq 1$ for an owned branch, from Chapter 5.

(ii) By (i) each side has length at least 1, so the three centres are pairwise distinct; and the constraints $\kappa_j(x) \geq 0$ say exactly that the triple is non-obtuse (Definition 9.14, now justified by distinctness). By Lemma 1.28 of Chapter 1 the three centres are therefore not collinear. This applies at every feasible point, not only at the configuration of (C), because non-obtuseness is one of the constraints. Noncollinearity gives $\det(P_2, P_3) \neq 0$, hence $D(x) \neq 0$; combined with the feasibility constraint $D(x) \geq 0$ this gives $D(x) > 0$, and then $D(x) = |\det(P_2, P_3)| = 2\,\mathrm{Area}$.

(iii) and (iv). By (i) all sides are at least 1, by feasibility the triangle is non-obtuse, and by (ii) its doubled area $D(x)$ is less than 1. The subcritical triangle lemma of Chapter 2 applies: all sides are less than $\sqrt{2}$ and the triangle is strictly acute. Since $P_1 = 0$, the sides P_1P_2 and P_1P_3 have lengths $\|P_2\|$ and $\|P_3\|$. Strict acuteness is the statement $\kappa_j(x) > 0$ for all j, by Proposition 1.25 again, the centres being pairwise distinct by (ii). □

There is an asymmetry in Lemma 9.15. The problem imposes the non-strict conditions $D \geq 0$ and $\kappa_j \geq 0$; the lemma derives the strict conditions $D > 0$ and $\kappa_j > 0$. The non-strict versions make X closed, which with boundedness gives compactness; the strict versions make those constraints slack at every feasible point, so that a descent step may ignore them. Both are needed. Had strict inequalities been written into the definition of X, the set would in general fail to be closed and the extreme value theorem would be unavailable.

Theorem 9.16 (Existence and geometry of the minimum)**.** *Assume (C) and freeze the data as in Definition 9.4. Then the feasible set X is nonempty, closed and bounded, the minimum of D over X is attained, and every minimiser x_* satisfies*

$$0 < D(x_*) \leq D_0 < 1$$

and has strictly acute centre triangle.

Proof. Nonempty. Let $x^{(0)}$ be the configuration of (C), normalised by translating so that $P_1 = 0$; by Lemma 9.13 this changes no branch value, no κ_j and not D. Its branch inequalities hold because the branches were constructed from it in step (1) of the freeze; its κ_j are nonnegative because (C) assumes the centre triangle non-obtuse; and $D(x^{(0)}) = D_0$, so $0 \leq D(x^{(0)}) \leq D_0$. Hence $x^{(0)} \in X$.

Bounded. Let $x = (P_2, P_3) \in X$. By Lemma 9.15(iii), $\|P_2\| < \sqrt{2}$ and $\|P_3\| < \sqrt{2}$, so in the block inner product of $\mathbb{R}^4$,

$$\|x\|^2 = \|P_2\|^2 + \|P_3\|^2 < 2 + 2 = 4.$$

Thus X is contained in the closed ball of radius 2 about the origin of $\mathbb{R}^4$.

Closed. X is cut out by the eight non-strict inequalities $c_1 \geq 0, \ldots, c_8 \geq 0$ listed after Definition 9.14. Each c_j is a polynomial in the four coordinates of x — inner products and determinants of the blocks are polynomials — hence continuous by Chapter 7. A set cut out by finitely many non-strict inequalities between continuous functions is closed, again by Chapter 7. So X is closed.

Attained. X is nonempty, closed and bounded in $\mathbb{R}^4$, and D is continuous on it (a polynomial). By the extreme value theorem of Chapter 7, established there from Bolzano–Weierstrass applied coordinatewise, D attains a minimum on X at some $x_* \in X$.

Geometry at the minimiser. x_* is feasible, so Lemma 9.15 applies to it: $0 < D(x_*) \leq D_0 < 1$ and the centre triangle is strictly acute. □

Only one minimiser is needed; uniqueness is neither claimed nor used.

Definition 9.17 (Convention: the minimising configuration)**.** From here to the end of Chapter 12, *the minimising configuration* means one fixed minimiser x_* furnished by Theorem 9.16, chosen once. The symbols $P_1 = 0, P_2, P_3$, the edges d_i, the doubled area D and the point x refer to its data. The frames F_i, the normals n_i, the thresholds H_i, the owners, the sign σ and the number D_0 remain the frozen data of (C).

Corollary 9.18 (The minimiser is a minimal counterexample)**.** *At the minimising configuration the three unit squares $Q(F_i, P_i)$ have pairwise disjoint interiors, and their centre triangle is strictly acute with area $D/2 < \frac{1}{2}$. Moreover no configuration satisfying the frozen system, non-obtuseness and $0 \le D \le D_0$ has strictly smaller doubled area.*

Proof. Disjointness of the interiors is Lemma 9.9 applied to the minimising centres, which satisfy (9.1); acuteness and $0 < D \le D_0 < 1$ are Theorem 9.16; the area is $D/2$ by Lemma 9.15(ii); and the last clause is the definition of a minimiser. □

Minimality here is relative to X, generally a proper subset of the set of admissible configurations with these frames (Remark 9.10). There may be admissible triples with these frames and smaller centre-triangle area, failing one of the three frozen inequalities. This is harmless, since the contradiction at the end is drawn inside X.

9.7 Graphs, degrees, and one counting fact

The proof of complete contact needs some graph vocabulary, which Part IA does not supply, and one counting fact.

Definition 9.19. A *simple graph* $G = (V, E)$ consists of a finite set V, whose elements are *vertices*, together with a set E of *edges*, each edge being a two-element subset of V. Vertices u and v are *adjacent* if $\{u, v\} \in E$. The *degree* $\deg(v)$ of a vertex v is the number of edges containing v. The graph on three vertices with all three possible edges is written K_3 and called *complete*.

A simple graph has no repeated edges and no edge from a vertex to itself, and its edges are unordered: $\{u, v\}$ and $\{v, u\}$ are the same edge. On three vertices there are three possible edges, so $|E| \le 3$ always.

Lemma 9.20 (Handshake lemma)**.** *For every finite simple graph $G = (V, E)$,*

$$\sum_{v \in V} \deg(v) \;=\; 2|E|.$$

Proof. Count the pairs (v, e) with $v \in V$, $e \in E$ and $v \in e$, in two ways. Grouping by v, the number of such pairs with first coordinate v is by definition $\deg(v)$, so the total is $\sum_v \deg(v)$. Grouping by e, each edge is a two-element set and so lies in exactly two such pairs, giving $2|E|$. Both counts count the same finite set. □

Corollary 9.21. *A simple graph on three vertices with at most two edges has a vertex of degree at most 1. Equivalently: if every vertex of a simple graph on three vertices has degree at least 2, then the graph is K_3.*

Proof. We prove the second form; the first is its contrapositive together with the fact that K_3 has three edges. Suppose every vertex has degree at least 2. Then $\sum_v \deg(v) \ge 6$, so $|E| \ge 3$ by Lemma 9.20. But a graph on three vertices has at most $\binom{3}{2} = 3$ edges, so $|E| = 3$ and all three possible edges are present, i.e. $G = K_3$. □

No other graph theory is used. Chapter 10 introduces the *owner tournament*, whose edges are directed, each pointing away from the owner of the corresponding branch; the active graph of this section is a different object, undirected, its edges recording which branches are active.

0 edges, $(0,0,0)$ 1 edge, $(1,1,0)$ 2 edges, $(2,1,1)$ 3 edges, $(2,2,2)$

Figure 9.3: The four simple graphs on three vertices up to isomorphism, with their degree sequences. Only the last has all degrees at least 2, as in Corollary 9.21.

9.8 The descent toolkit

Definition 9.22 (Active branch, active graph, active degree)**.** At a feasible configuration, branch i is *active* if $\langle d_i, n_i\rangle = H_i$ and *slack* if $\langle d_i, n_i\rangle > H_i$. Branch i is *incident* to the two vertices P_i and P_{i+1}. The *active graph* is the simple graph on the vertex set $\{P_1, P_2, P_3\}$ whose edges are the pairs $\{P_i, P_{i+1}\}$ for which branch i is active, and the *active degree* of a vertex is its degree in the active graph.

Lemma 9.23 (Local data at a vertex)**.** *Fix $j \in \{1,2,3\}$, indices modulo 3. The branches incident to P_j are branch j and branch $j+2$; the remaining branch, branch $j+1$, involves only P_{j+1} and P_{j+2}, and its edge $d_{j+1} = P_{j+2} - P_{j+1}$ is the side of the triangle opposite P_j. As inequalities on the position of P_j alone, with the other two centres regarded as fixed, and writing Q for the other endpoint of the branch in question,*

$$\boxed{\begin{array}{rlll} \textit{branch } j: & \big\langle P_j - P_{j+1},\, -n_j\big\rangle & \geq\ H_j, & Q = P_{j+1}, \\ \textit{branch } j+2: & \big\langle P_j - P_{j+2},\, n_{j+2}\big\rangle & \geq\ H_{j+2}, & Q = P_{j+2}. \end{array}}$$

The vectors $-n_j$ and n_{j+2} are called the locally oriented normals at P_j*: each is the frozen normal of an incident branch, or its negative, whichever writes the frozen inequality in the displayed form.*

Proof. Branch i is $\langle d_i, n_i\rangle \geq H_i$ with $d_i = P_{i+1} - P_i$, so it involves only the centres P_i and P_{i+1}; hence the branches involving P_j are those with $i = j$ and with $i+1 = j$, that is $i = j$ and $i = j-1 = j+2$, and the remaining one is $i = j+1$, whose edge $d_{j+1} = P_{j+2} - P_{j+1}$ joins the two vertices other than P_j.

For branch j: $\langle d_j, n_j\rangle = \langle P_{j+1} - P_j, n_j\rangle = \langle P_j - P_{j+1}, -n_j\rangle$, giving the first line. For branch $j+2$: since indices are modulo 3, $d_{j+2} = P_{j+3} - P_{j+2} = P_j - P_{j+2}$, so $\langle d_{j+2}, n_{j+2}\rangle = \langle P_j - P_{j+2}, n_{j+2}\rangle$, giving the second line. □

Lemma 9.24 (Single-vertex variation)**.** *Fix a configuration P_1, P_2, P_3, an index j, and a vector $v \in \mathbb{R}^2$. For $t \in \mathbb{R}$ let the varied configuration be the one obtained by replacing P_j with $P_j + tv$ and leaving the other two centres fixed (followed, if $j = 1$, by the translation that restores $P_1 = 0$, which changes nothing by Lemma 9.13). Write $g_j = J_\sigma d_{j+1}$. Then, as functions of t:*

- *(i) the value of branch $j+1$ is constant;*
- *(ii) the values of branches j and $j+2$ are affine, with t-derivatives $\langle -n_j, v\rangle$ and $\langle n_{j+2}, v\rangle$ respectively;*
- *(iii) with the other two centres fixed, the doubled area is an affine function of the position of P_j on all of $\mathbb{R}^2$, with constant gradient g_j; in particular $D(t) = D + t\,\langle g_j, v\rangle$;*
- *(iv) $\|g_j\| = \|d_{j+1}\| \ \geq\ H_{j+1} \ \geq\ 1$, provided the original configuration satisfies branch $j+1$;*
- *(v) in particular, taking $v = -g_j$ gives $D(t) = D - t\|g_j\|^2$.*

Proof. (i) By Lemma 9.23, branch $j+1$ is a function of P_{j+1} and P_{j+2} only, and neither moves.

(ii) By Lemma 9.23 the value of branch j is $\langle P_j - P_{j+1}, -n_j\rangle$, which under $P_j \mapsto P_j + tv$ becomes $\langle P_j - P_{j+1}, -n_j\rangle + t\langle -n_j, v\rangle$: affine in t with the stated derivative. Likewise for branch $j+2$ with normal n_{j+2}.

(iii) This is the vertex-gradient theorem 6.22 of Chapter 6, proved there for each of the three vertices: the gradient at a vertex is J_σ of the opposite edge, and does not depend on where the moving vertex is. The displayed formula is then the affine identity $f(X + tv) = f(X) + t\langle g, v\rangle$ of

Chapter 6. Affineness holds on all of $\mathbb{R}^2$, not only along the line $t \mapsto P_j + tv$; Step 3 of Theorem 9.26 uses this at the other two vertices.

(iv) J_σ is a rotation, hence an isometry, so $\|g_j\| = \|d_{j+1}\|$; and $\|d_{j+1}\| \geq \langle d_{j+1}, n_{j+1}\rangle \geq H_{j+1} \geq 1$ by Cauchy–Schwarz, branch $j+1$, and the threshold bound. The bound thus comes from branch $j+1$, which by (i) the motion of P_j leaves fixed.

(v) Substitute $v = -g_j$ in (iii), or quote the identity $f(x - tg) = f(x) - t\|g\|^2$ of Chapter 6. □

Chapter 7 proves the first-order principle for this problem; we apply it to single-vertex motions.

Corollary 9.25 (First-order feasibility at a vertex)**.** *Let the configuration be the minimising one, fix a vertex index j, and let $v \in \mathbb{R}^2$ satisfy*

$$\langle n, v\rangle \ \geq \ 0 \qquad \textit{for every locally oriented normal } n \textit{ at } P_j \textit{ of an active branch.}$$

Then $\langle g_j, v\rangle \geq 0$, where $g_j = J_\sigma d_{j+1}$. Equivalently: at the minimising configuration there is no vertex motion that weakly relaxes every active branch at that vertex and strictly decreases the doubled area.

Proof. The frozen problem is the eight-constraint problem of Example 7.26, so we check the three hypotheses of Theorem 7.36 at the minimiser x_*. (a) D is a polynomial and is differentiable with gradient g (Chapter 6). (b) Every constraint is a polynomial, hence continuous. (c) By Lemma 9.15(ii),(iv) we have $\kappa_1, \kappa_2, \kappa_3 > 0$ and $D > 0$ at x_*, so $c_4, \ldots, c_7$ are slack; the only constraints that can be active are c_1, c_2, c_3, which are affine with gradients a_1, a_2, a_3, and $c_8 = D_0 - D$, which is of objective-bound type. Thus $\mathrm{Aff}(x_*)$ is the set of active branches and $\mathrm{Obj}(x_*) \subseteq \{8\}$, and hypothesis (c) holds. The strict inequalities of Lemma 9.15 put the four non-affine constraints in the slack column, as required by Remark 7.32 of Chapter 7.

Now translate. By the vertex dictionary, Proposition 6.12, moving P_j with velocity v is the velocity

$$v_* = (v, 0), \qquad v_* = (0, v), \qquad v_* = (-v, -v) \in \mathbb{R}^4$$

for $j = 2$, $j = 3$, $j = 1$ respectively. Pairing with the constant branch gradients, in the case $j = 2$,

$$\langle a_1, v_*\rangle = \langle n_1, v\rangle, \qquad \langle a_2, v_*\rangle = \langle -n_2, v\rangle, \qquad \langle a_3, v_*\rangle = 0,$$

and n_1, $-n_2$ are precisely the locally oriented normals at P_2 of the incident branches $2 + 2 = 1$ and 2 furnished by Lemma 9.23, while branch 3 is the opposite one. The cases $j = 3$ and $j = 1$ are the same computation; the case $j = 1$ is carried out in Example 6.28 of Chapter 6. That same example records $\langle g, v_*\rangle = \langle g_j, v\rangle$, and for $j = 2, 3$ this is immediate from $g = (J_\sigma d_3, J_\sigma d_1)$.

The hypothesis of the corollary therefore says $\langle a_i, v_*\rangle \geq 0$ for every $i \in \mathrm{Aff}(x_*)$, that is, $v_* \in V(x_*)$; note that the opposite branch contributes $\langle a_{j+1}, v_*\rangle = 0$ and so imposes nothing whether active or not. Theorem 7.36 gives $\langle g, v_*\rangle \geq 0$, i.e. $\langle g_j, v\rangle \geq 0$. □

Two features of the bookkeeping are used again in Chapter 10. The constraint $D \leq D_0$ never obstructs a descent direction, active or not, since any motion that decreases D preserves it (Lemma 7.31); and an active affine constraint with $\langle a_i, v\rangle \geq 0$ survives along the whole ray, not only for small t (Lemma 7.30(i)). With the slackness of $c_4, \ldots, c_7$, these are why the first-order analysis sees only the three branch constraints, and why the Farkas lemma of Chapter 10 is applied with the three gradients a_1, a_2, a_3 and not with eight.

9.9 Complete contact

Theorem 9.26 (Complete contact)**.** *At the minimising configuration of Theorem 9.16, all three frozen branches are active:*

$$\langle d_i, n_i\rangle \ = \ H_i \qquad (i = 1, 2, 3).$$

Proof. Form the active graph of Definition 9.22 at the minimising configuration. We show that no vertex has active degree 0 or 1. Given that, every vertex has active degree at least 2, so by Corollary 9.21 the active graph is K_3; that is, all three branches are active, which is the assertion.

Suppose then that some vertex $P = P_j$ has active degree at most 1. Write $g = g_j = J_\sigma d_{j+1}$ for the area gradient at that vertex, so $\|g\| \geq 1$ by Lemma 9.24(iv). There are two cases, drawn in Figure 9.4.

Case A: active degree 0. Neither branch incident to P_j is active, so the hypothesis of Corollary 9.25 is vacuous: there are no locally oriented active normals to test against, and every v qualifies. Take $v = -g$. The corollary gives $\langle g, -g\rangle \geq 0$, that is

$$0 \;\leq\; \langle g, v\rangle = -\|g\|^2 \;\leq\; -1,$$

which is absurd. So active degree 0 is impossible. (Concretely, the motion $P_j \mapsto P_j - tg$ produces for small $t > 0$ a feasible configuration with $D(t) = D - t\|g\|^2 < D$, by Lemma 9.24(v), contradicting minimality; the corollary states this in first-order form.)

Case B: active degree 1. Let the unique active branch at $P = P_j$ be locally oriented as in Lemma 9.23, with locally oriented normal n (a frozen normal or its negative, so a unit vector), frozen threshold $H \geq 1$, and other endpoint Q (one of P_{j+1}, P_{j+2}). Activity says

$$\langle P - Q,\, n\rangle = H.$$

Step 1 (first-order condition). Let $v \in \mathbb{R}^2$ satisfy $\langle n, v\rangle \geq 0$. Since n is the only locally oriented active normal at P, the hypothesis of Corollary 9.25 holds for v, and its conclusion gives $\langle g, v\rangle \geq 0$. That is:

$$\langle g, v\rangle \geq 0 \qquad \text{for every } v \text{ with } \langle n, v\rangle \geq 0.$$

This is the single-active-constraint case, Corollary 7.38 of Chapter 7, read at one vertex.

Step 2 (half-plane duality). By the half-plane lemma 8.12 of Chapter 8 — if n is a unit vector and $\langle g, v\rangle \geq 0$ for every v in the half-plane $\langle n, v\rangle \geq 0$, then $g = \mu n$ with $\mu = \langle g, n\rangle \geq 0$ — we get $g = \mu n$ with $\mu \geq 0$. Since $\|n\| = 1$ we have $\|g\| = |\mu| = \mu$, and $\|g\| \geq 1$, so

$$\mu \;\geq\; 1.$$

Step 3 (the affine difference identity). Let $\varphi(X)$ denote the doubled signed area of the triple obtained from the minimising configuration by replacing $P = P_j$ with the point X, the other two vertices being held fixed. By Lemma 9.24(iii), φ is affine with constant gradient g, and $\varphi(P) = D$. Moreover

$$\varphi(P_{j+1}) = \varphi(P_{j+2}) = 0,$$

because placing X at another vertex makes two of the three points coincide, so the triple is degenerate and the determinant vanishes. By the exact difference identity for affine functions (Chapter 6), for either other vertex Q',

$$\langle g,\, P - Q'\rangle = \varphi(P) - \varphi(Q') = D - 0 = D.$$

Both other vertices give the same value, so it does not matter which one the active branch joins to P. The identity is exact, with no remainder term, because φ is affine.

Step 4 (the contradiction). Put $r = P - Q$, where Q is the other endpoint of the active branch. Activity gives $\langle n, r\rangle = H$, and Steps 2 and 3 give

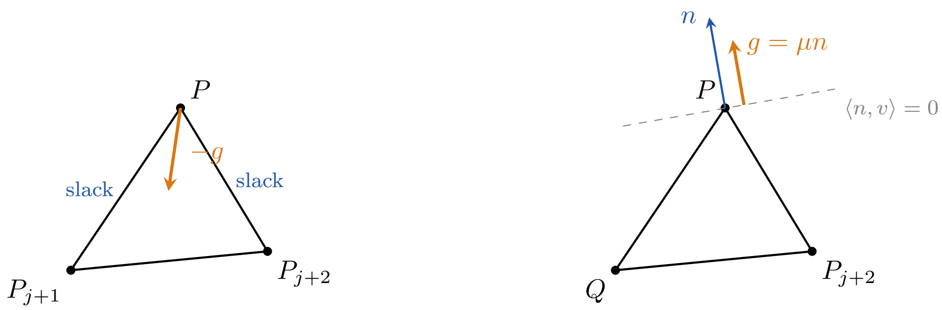
$$D = \langle g, r\rangle = \mu\,\langle n, r\rangle = \mu H \;\geq\; 1,$$

since $\mu \geq 1$ and $H \geq 1$. But $D \leq D_0 < 1$ at the minimising configuration by Theorem 9.16. This contradiction rules out active degree 1.

Neither active degree 0 nor active degree 1 occurs, so every vertex has active degree 2, and by Corollary 9.21 the active graph is K_3: all three branches are active. □

Example 9.27 (The construction in numbers)**.** Take all three frames standard and the frozen data

$$n_1 = (1,0), \quad n_2 = (-1,0), \quad n_3 = (0,-1), \quad H_1 = H_2 = H_3 = 1, \quad \sigma = +1,$$



active degree 0:
move down the area gradient

active degree 1:
the gradient lies on the normal ray

Figure 9.4: The two ways contact could fail at a vertex P. Left: with no active incident branch, P moves along $-g$ and the doubled area strictly decreases. Right: with exactly one active incident branch, first-order minimality over the half-plane $\langle n, v\rangle \geq 0$ forces the area gradient onto the normal ray, $g = \mu n$ with $\mu \geq 1$, and then $D = \mu H \geq 1$, contradicting $D < 1$.

so that $J_\sigma(a, b) = (-b, a)$. Start from

$$P_1 = (0, 0), \qquad P_2 = (2, 0), \qquad P_3 = (\tfrac{1}{2}, 3).$$

As in Remark 9.8, this configuration does not satisfy (C); the computation below checks identities, not hypotheses.

Branch values. $d_1 = (2, 0)$, $d_2 = (-\frac{3}{2}, 3)$, $d_3 = (-\frac{1}{2}, -3)$, so

$$\langle d_1, n_1\rangle = 2, \qquad \langle d_2, n_2\rangle = \tfrac{3}{2}, \qquad \langle d_3, n_3\rangle = 3,$$

all strictly greater than the corresponding $H_i = 1$. Every branch is slack, so every vertex has active degree 0, and we are in Case A of Theorem 9.26 at every vertex.

The gradient at P_3. The edge opposite P_3 is $d_{3+1} = d_1 = (2, 0)$, so

$$g_3 = J_\sigma d_1 = J_\sigma(2, 0) = (0, 2), \qquad \|g_3\| = 2 = \|d_1\| \;\geq\; H_1 = 1,$$

confirming Lemma 9.24(iv): the bound comes from branch 1, which moving P_3 does not disturb.

Descent. Move P_3 along $-g_3$: $P_3(t) = (\frac{1}{2}, 3) - t(0, 2) = (\frac{1}{2}, 3 - 2t)$. Then

$$D(t) = \sigma \det\bigl((2, 0), (\tfrac{1}{2}, 3 - 2t)\bigr) = 2(3 - 2t) = 6 - 4t = D - t\|g_3\|^2,$$

confirming Lemma 9.24(v) with $D = 6$ and $\|g_3\|^2 = 4$.

What the branches do along the motion. Branch 1 does not involve P_3 and keeps the value 2 (Lemma 9.24(i) with $j = 3$: the opposite branch is branch 1). Branch 2 has value $\langle P_3(t) - P_2, n_2\rangle = \langle(-\frac{3}{2}, 3 - 2t), (-1, 0)\rangle = \frac{3}{2}$ for all t: the velocity $-g_3 = (0, -2)$ is orthogonal to n_2, so by Lemma 9.24(ii) the derivative is 0. Branch 3 has value $\langle P_1 - P_3(t), n_3\rangle = 3 - 2t$, which reaches $H_3 = 1$ at $t = 1$. Descent therefore stops at

$$P_3 = (\tfrac{1}{2}, 1), \qquad D = 2.$$

Strict acuteness is preserved throughout: $\kappa_1 = \langle P_2, P_3(t)\rangle = 1$, $\kappa_2 = \|P_2\|^2 - \kappa_1 = 3$ and $\kappa_3 = \|P_3(t)\|^2 - \kappa_1 = \frac{1}{4} + (3 - 2t)^2 - 1$, which for $0 \leq t \leq 1$ is at least $\frac{1}{4}$. All three are strictly positive, in accordance with Lemma 9.15(iv).

Case B at the stopping point. At $P_3 = (\frac{1}{2}, 1)$ the vertex P_3 has active degree 1: branch 3 is active, branch 2 has value $\frac{3}{2} > 1$. Apply Lemma 9.23 with $j = 3$: the incident branches are 3 and $3 + 2 = 2$, and the active one, branch 3, reads

$$\langle P_3 - P_1,\, -n_3\rangle \;\geq\; H_3, \qquad -n_3 = (0, 1), \quad Q = P_1.$$

So the locally oriented active normal is $n = (0,1)$, and with $r = P_3 - P_1 = (\frac{1}{2}, 1)$ we have $\langle n, r\rangle = 1 = H$, confirming activity. Note that r is not parallel to n: activity constrains one coordinate only. Now check the two conclusions of Steps 2 and 4:

$$g_3 = (0,2) = 2\,(0,1) = \mu n \quad \text{with } \mu = 2 \geq 1, \qquad D = \mu H = 2 \cdot 1 = 2,$$

matching the direct computation $D = \sigma \det\big((2,0), (\frac{1}{2}, 1)\big) = 2$. Finally check Step 3 against the other vertex:

$$\langle g_3,\, P_3 - P_2\rangle = \big\langle (0,2),\, (\tfrac{1}{2}, 1) - (2,0)\big\rangle = \big\langle (0,2), (-\tfrac{3}{2}, 1)\big\rangle = 2 = D.$$

Both other vertices give D, as Step 3 asserts.

There is no contradiction here, since $D = 2 \geq 1$; the argument of Theorem 9.26 applies only under the cap $D \leq D_0 < 1$, which this configuration violates. The example checks the identities the proof uses.

9.10 What the next chapter uses

Chapter 10 uses the first-order principle with the whole configuration $x \in \mathbb{R}^4$ moving at once, rather than one vertex at a time. The whole-ray property of active affine constraints (Lemma 7.30(i)), available because the branch functions b_i are linear, is hypothesis (ii) of the Farkas lemma of Chapter 8.

Complete contact replaces each occurrence of $\langle d_i, n_i\rangle$ by the constant H_i, and that substitution turns the vector multiplier equation into the sine system: dotting the multiplier identity with n_2, say, produces $H_2 = \lambda_1 s_1 + \lambda_3 s_2$, an equation between known thresholds and unknown multipliers. Without contact the left-hand sides would be the unknown numbers $\langle d_i, n_i\rangle$, and Chapters 11–12 would have no numerical data to work with. The area identity

$$2D = \lambda_1 H_1 + \lambda_2 H_2 + \lambda_3 H_3$$

does not need contact, however: a branch that happened to be slack would receive multiplier $\lambda_i = 0$ (Remark 7.39), so that $\lambda_i b_i(x) = \lambda_i H_i$ term by term, with equality when the branch is active and both sides zero when it is slack. Complete contact removes the need for that complementary-slackness bookkeeping; it is not what makes the identity true.

Exercises

Exercise 9.1 (The equality configuration)**.** Let S_1, S_2, S_3 be axis-parallel unit squares centred at $(0,0)$, $(1,0)$ and $(0,1)$.

(a) Verify that the three interiors are pairwise disjoint, and compute σ and D.

(b) For each pair, list every unit vector that is an axis of one of the two frames and yields a valid branch for that pair, and compute its threshold.

(c) Show that each branch listed in (b) is active.

(d) Deduce that the conclusion of Theorem 9.26 is compatible with $D = 1$, and explain which step of the proof of Theorem 9.26 fails if the constraint $D \leq D_0$ is weakened to $D \leq 1$.

Exercise 9.2 (Degrees, in general and in the active graph)**.** (a) Prove that a finite simple graph on n vertices in which every vertex has degree at least $n-1$ is complete, and check that Corollary 9.21 is the case $n = 3$. (Handshake plus the bound $|E| \leq \binom{n}{2}$.)

(b) Show that "at least $n-1$" cannot be weakened to "at least $n-2$": for every $n \geq 3$ exhibit a non-complete simple graph on n vertices with all degrees at least $n-2$.

(c) Fix the frozen data $n_1 = (1,0)$, $n_2 = (-1,0)$, $n_3 = (0,-1)$, $H_1 = H_2 = H_3 = 1$, $\sigma = +1$, and call a triple $P_1 = 0, P_2, P_3$ *admissible* if it satisfies the three branch inequalities, the three non-obtuseness inequalities and $D \geq 0$, that is, every constraint of Definition 9.14 except the cap. For each of the four degree sequences of Figure 9.3, exhibit an admissible triple whose active graph realises it. (For $(1,1,0)$ try $P_2 = (1,1)$, $P_3 = (-1,2)$.)

(d) Explain why (c) does not contradict Theorem 9.26, and compute D for each of the four triples.

Exercise 9.3 (What each constraint does)**.** Fix the frozen data $n_1 = (1,0)$, $n_2 = (-1,0)$, $n_3 = (0,-1)$, $H_1 = H_2 = H_3 = 1$, $\sigma = +1$, and fix any $D_0 \in (0,1)$.

(a) Show that $P_1 = 0$, $P_2 = (R,0)$, $P_3 = (0,R)$ satisfies the three branch inequalities and the three non-obtuseness inequalities for every $R \geq 1$, and deduce that the set defined by all the constraints of Definition 9.14 except $D \leq D_0$ is unbounded.

(b) Show that if instead the three branch inequalities are dropped, keeping $\kappa_j \geq 0$ and $0 \leq D \leq D_0$, then the infimum of D over the remaining set is 0 and is attained, at $P_2 = P_3 = 0$.

(c) Conclude that the cap is what bounds X, and that the branch inequalities keep a minimiser nondegenerate: in (b) the minimiser is a single point counted three times, and yields no contradiction. Then explain why hypothesis (C) is still needed for a third reason, visible in neither (a) nor (b): without a point known to satisfy all eight constraints, X may be empty and the minimisation vacuous.

Exercise 9.4 (Descent, computational)**.** With the frozen data of the previous exercise and the configuration $P_1 = (0,0)$, $P_2 = (3,0)$, $P_3 = (0,2)$:

(a) compute all three branch values and the active degree of each vertex;

(b) compute g_1, g_2, g_3 and verify $\|g_j\| = \|d_{j+1}\|$ in each case;

(c) descend from P_2 along $-g_2$, verify $D(t) = D - t\|g_2\|^2$ by direct computation, and find the exact value of t at which the configuration first acquires an active branch. Show that two branches become active there simultaneously, so that P_2 jumps from active degree 0 to active degree 2;

(d) at that value of t, identify the two locally oriented active normals at P_2 using Lemma 9.23, and observe that they coincide. Deduce that $g_2 = \mu n$ with $\mu = 2$, and verify the Step 4 identity $D = \mu H$ directly. Why is this configuration not a counterexample to Theorem 9.26?

(e) show that the coincidence in (d) is forced by the data rather than by the descent: since $n_2 = -n_1$ here, branches 1 and 2 have equal values at every configuration with $\langle P_3, n_1 \rangle = 0$, and $P_3 = (0,2)$ is such a configuration. Then replace $P_3 = (0,2)$ by $P_3 = (1,2)$, keeping $P_1 = (0,0)$ and $P_2 = (3,0)$, and repeat (c) and (d): exactly one branch now becomes active, g_2 is not a nonnegative multiple of the locally oriented active normal there, and a v with $\langle n, v \rangle \geq 0$ and $\langle g_2, v \rangle < 0$ exists. Find one, and say what Corollary 9.25 concludes about that configuration.

Exercise 9.5 (Activity versus tangency)**.** (a) Exhibit two unit squares with disjoint interiors, together with a valid branch for them whose normal is an axis of one of the two frames and which is active, such that the two boundaries are at distance at least 10.

(b) Conversely, prove that if two unit squares $Q(F,P)$ and $Q(G,Q)$ have disjoint interiors and their boundaries meet, then every valid branch for that pair is active. (Use the description of overlap in terms of the zonogon $Z = Q(F,0) + Q(G,0)$ and its support function, from Chapters 4 and 5.)

(c) Deduce that "the squares touch" implies "every branch is active" but not conversely, and say why the failure of the converse is harmless for the argument of this chapter.

Exercise 9.6 (First-order feasibility from scratch)**.** Prove Corollary 9.25 directly, without quoting Theorem 7.36: state the eight constraints of the frozen problem, and for each one give the reason it continues to hold along the varied configuration for small $t > 0$. Then classify the eight: which require strictness at the minimising configuration, which require only continuity, and which require neither? Compare the classification with the checklist of Remark 7.40.

Exercise 9.7 (Two active branches at a vertex; preview of Chapter 10)**.** Let n, n' be linearly independent unit vectors in $\mathbb{R}^2$ and let $g \in \mathbb{R}^2$ satisfy $\langle g, v \rangle \geq 0$ for every v with $\langle n, v \rangle \geq 0$ and $\langle n', v \rangle \geq 0$. Prove that $g = \alpha n + \beta n'$ with $\alpha, \beta \geq 0$. (Hint: write $g = \alpha n + \beta n'$, which is possible because n, n' are a basis of $\mathbb{R}^2$, and test the hypothesis against the vector v with $\langle n, v \rangle = 0$, $\langle n', v \rangle = 1$ and against the vector v with $\langle n, v \rangle = 1$, $\langle n', v \rangle = 0$. Both exist: the map $v \mapsto (\langle n, v \rangle, \langle n', v \rangle)$ is linear $\mathbb{R}^2 \to \mathbb{R}^2$ and injective, since $\langle n, v \rangle = \langle n', v \rangle = 0$ forces v orthogonal to a basis, hence $v = 0$; so it is onto.) Then, granting Theorem 9.26, show that this applies at every vertex of the minimising configuration with its two locally oriented active normals, provided those are linearly independent. What can be said if they are parallel, and what does Example 9.7 suggest about when that happens?

Chapter 10

Multipliers, the Reciprocal Diagram, and the Sine System

What this chapter uses

- From Chapter 2: the quarter turn J_σ, with $J_\sigma^2 = -I$, $\langle J_\sigma u, v\rangle = \sigma \det(u, v)$, and the skewness $\langle u, J_\sigma v\rangle = -\langle J_\sigma u, v\rangle$, which follows from antisymmetry of the determinant.
- Also from Chapter 2, four further properties of J_σ, used in §10.4 and §10.6: from Proposition 2.8(iii)–(v), $\|J_\sigma u\| = \|u\|$, $\det(J_\sigma u, J_\sigma v) = \det(u, v)$ and $J_{-\sigma} = -J_\sigma$; and from Definition 2.7, the description of J_σ as the rotation of the plane through $\sigma\pi/2$.
- From Chapter 5: the notion of an owned branch and its owner; the axes of a frame; the owned-branch threshold $H(t) = (1+\cos t+\sin t)/2 \geq 1$, so that every frozen threshold satisfies $H_i \geq 1$.
- From Chapter 6: the block inner product on $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$; the distinction between linear and affine; the gradient of the doubled signed area; Euler's identity in degrees 2 and 1; and the fact that $\langle g, v\rangle < 0$ forces strict decrease of the objective along the ray $t \mapsto x + tv$ for all small $t > 0$.
- From Chapter 7: active versus slack constraints; the fact that a strict inequality between continuous functions is an open condition and therefore persists under small perturbations; and the no-feasible-descent-direction principle, together with the eight-constraint list of the minimisation problem.
- From Chapter 8: the finite Farkas lemma, in the form restated as Theorem 10.1 below.
- From Chapter 9: the freeze; the existence of a minimising configuration; complete contact.

Standing hypotheses

Theorem 1.1 is assumed false. Fix a counterexample: unit squares $S_i = Q(F_i, P_i)$, $i = 1, 2, 3$, with pairwise disjoint interiors, non-obtuse centre triangle, and doubled area $D_0 < 1$. The freeze of Chapter 9 selects, once and for all, one owned separating branch for each of the three pairs. The frames F_1, F_2, F_3, the unit normals n_1, n_2, n_3, the thresholds H_1, H_2, H_3, the owners, and the orientation sign $\sigma \in \{-1, +1\}$ are frozen constants; only the centres vary.

We normalise $P_1 = 0$ and write $x = (P_2, P_3) \in \mathbb{R}^4$ for the configuration point. The cyclic edges are

$$d_1 = P_2 - P_1, \qquad d_2 = P_3 - P_2, \qquad d_3 = P_1 - P_3, \qquad \text{so } d_1 + d_2 + d_3 = 0,$$

and the *objective* is the doubled area

$$D(x) = \sigma \det(d_1, d_2) = \sigma \det(P_2, P_3),$$

the two expressions agreeing because $P_1 = 0$ makes $d_1 = P_2$ and $d_2 = P_3 - P_2$, so that $\det(d_1, d_2) = \det(P_2, P_3 - P_2) = \det(P_2, P_3)$. We write $D = D(x)$ for its value at the configuration under discussion. (Chapter 7 writes f for the objective; in this chapter f_i denotes the forces of Definition 10.7.) Throughout this chapter x is a fixed minimiser produced by the existence lemma of Chapter 9, so that

$$0 < D \le D_0 < 1, \qquad \text{the centre triangle is acute,}$$

$$\langle d_i, n_i \rangle = H_i \quad (i = 1, 2, 3),$$

the last line being complete contact. Every $H_i \ge 1$, because every frozen branch is owned. All indices on $P, d, n, H, s, \lambda, f$ are read modulo 3 in $\{1, 2, 3\}$.

Throughout, "active branch" means equality in a chosen frozen linear inequality, not geometric tangency of squares; no square is moved or rotated.

Chapter 9 produced a geometric fact: at the minimiser all three frozen branches hold with equality. That statement is geometric; the remaining case analysis needs algebra. This chapter supplies the translation in three steps. First, the Farkas lemma turns first-order optimality into nonnegative multipliers $\lambda_1, \lambda_2, \lambda_3$, and Euler's identity turns those multipliers into the area identity $2D = \lambda_1 H_1 + \lambda_2 H_2 + \lambda_3 H_3$. Second, resolving the multiplier equation into its two planar components gives the reciprocal equations, a quarter-turned copy of the centre triangle pinned to the three normal rays. Third, dotting the reciprocal equations with the normals gives three scalar equations, the sine system, which is what Chapters 11 and 12 use. The chapter ends with the relabelling conventions, one easy case, and a classification of ownership patterns, all of which the next two chapters presuppose.

10.1 Checking the two Farkas hypotheses at the minimiser

Chapter 8 proved a theorem with two hypotheses; we check them at the minimiser.

Theorem 10.1 (Finite Farkas lemma; Chapter 8)**.** *Let $a_1, \ldots, a_m$ and g be vectors in $\mathbb{R}^N$. Suppose*
(i) there is a vector w with $\langle a_i, w \rangle > 0$ for every i;
(ii) every v with $\langle a_i, v \rangle \ge 0$ for all i satisfies $\langle g, v \rangle \ge 0$.
Then $g = \sum_{i=1}^{m} \lambda_i a_i$ for some $\lambda_1, \ldots, \lambda_m \ge 0$.

We apply it with $N = 4$ and $m = 3$. Hypothesis (ii) is the first-order optimality information; hypothesis (i) is a technical condition whose only role in the proof was to make the generated cone $\Gamma = \mathrm{cone}(a_1, \ldots, a_m)$ closed, so that a point outside it could be separated from it.

Definition 10.2 (Branch functions and their gradients; Proposition 6.15)**.** With $P_1 = 0$ and $x = (P_2, P_3) \in \mathbb{R}^4$, put

$$b_1(x) = \langle P_2, n_1\rangle, \qquad b_2(x) = \langle P_3 - P_2, n_2\rangle, \qquad b_3(x) = \langle -P_3, n_3\rangle,$$

so that $b_i(x) = \langle d_i, n_i\rangle$ and the three frozen inequalities of Chapter 9 read $b_i(x) \geq H_i$. Each b_i is linear in x, with constant gradient

$$a_1 = (n_1, 0), \qquad a_2 = (-n_2, n_2), \qquad a_3 = (0, -n_3) \ \in \ \mathbb{R}^4.$$

Linearity gives

$$\langle a_i, x\rangle = b_i(x) \qquad \text{for every } x \in \mathbb{R}^4. \tag{10.1}$$

For an affine function this is false: an affine b with $b(x) = \langle a, x\rangle + c$ satisfies $\langle a, x\rangle = b(x) - c$, and the constant c does not go away. Equation (10.1) depends on the normalisation $P_1 = 0$; it is used in Proposition 10.4(i) and in Theorem 10.6.

Lemma 10.3 (Gradient of the objective)**.** *The objective $D(x) = \sigma \det(P_2, P_3)$ has value D at the minimiser, and*

$$g = \nabla D(x) = \big(-J_\sigma P_3,\ J_\sigma P_2\big) = \big(J_\sigma d_3,\ J_\sigma d_1\big).$$

Proof. By $\langle J_\sigma u, v\rangle = \sigma \det(u, v)$ we may write $D(x) = \langle J_\sigma P_2, P_3\rangle$. As a function of P_3 alone this is linear with gradient $J_\sigma P_2$. As a function of P_2 alone, skewness gives $D(x) = \langle J_\sigma P_2, P_3\rangle = -\langle P_2, J_\sigma P_3\rangle$, which is linear in P_2 with gradient $-J_\sigma P_3$. By the block structure of the inner product on $\mathbb{R}^4$ these two planar gradients are the two blocks of g. Finally $P_1 = 0$ gives $d_1 = P_2$ and $d_3 = -P_3$, so $-J_\sigma P_3 = J_\sigma d_3$ and $J_\sigma P_2 = J_\sigma d_1$. □

This is the area-gradient lemma of Chapter 6 read in $\mathbb{R}^4$: the gradient in one vertex is the quarter turn of the opposite edge.

Proposition 10.4 (The two hypotheses hold at the minimiser)**.** *With a_1, a_2, a_3 as in Definition 10.2 and g as in Lemma 10.3, hypotheses (i) and (ii) of Theorem 10.1 hold at the minimiser x.*

Proof. (i): the strict dual vector. Take $w = x$ itself, the dilation direction. By linearity (10.1) and complete contact,

$$\langle a_i, x\rangle = b_i(x) = H_i \geq 1 > 0 \qquad (i = 1, 2, 3).$$

Since $P_1 = 0$, moving from x in the direction x scales the centre triangle about P_1, and dilating strictly increases every branch function; so the technical hypothesis of Chapter 8 holds without further construction.

(ii): first-order optimality. This is Theorem 7.36 of Chapter 7, whose hypotheses are met by the eight-constraint list of Example 7.26. The objective D is differentiable with gradient g; the three branch constraints are active at x and affine, with gradients a_1, a_2, a_3; the objective bound $D \leq D_0$ is of objective-bound type; and the remaining constraints — the three non-obtuseness functions κ_j and $D > 0$ — are slack at the minimiser by Theorem 9.16 of Chapter 9, and are continuous, being polynomials. Theorem 7.36 therefore gives $\langle g, v\rangle \geq 0$ for every v in

$$V(x) = \{v \in \mathbb{R}^4 : \ \langle a_i, v\rangle \geq 0 \text{ for } i = 1, 2, 3\},$$

which is hypothesis (ii). □

Remark 10.5. Linearity of the b_i gives feasibility of the branch constraints along the whole ray $x + tv$, $t \geq 0$, and not only for small t. Hence the set of admissible velocities is a cone, and the Farkas lemma applies where a purely local argument would not.

10.2 The multipliers and the area identity

Theorem 10.6 (Multipliers, and the area identity)**.** *At the minimiser there exist $\lambda_1, \lambda_2, \lambda_3 \geq 0$ with*

$$g = \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3,$$

and moreover

$$2D = \lambda_1 H_1 + \lambda_2 H_2 + \lambda_3 H_3. \tag{10.2}$$

Proof. The first assertion is Theorem 10.1 applied to a_1, a_2, a_3, g, whose hypotheses were verified in Proposition 10.4.

For the second: the objective $D(x) = \sigma \det(P_2, P_3)$ is homogeneous of degree 2 in x, since det is bilinear, and each b_i is homogeneous of degree 1, being linear. Euler's identity from Chapter 6 gives $\langle g, x\rangle = 2D(x) = 2D$, while $\langle a_i, x\rangle = b_i(x) = H_i$ by (10.1) and complete contact. Therefore

$$2D = \langle g, x\rangle = \Big\langle \sum_{i=1}^{3} \lambda_i a_i,\ x \Big\rangle = \sum_{i=1}^{3} \lambda_i \langle a_i, x\rangle = \lambda_1 H_1 + \lambda_2 H_2 + \lambda_3 H_3. \qquad \square$$

Definition 10.7 (Forces)**.** The *forces* of the minimising configuration are $f_i = \lambda_i n_i$ for $i = 1, 2, 3$. Thus $\|f_i\| = \lambda_i$, and f_i lies on the ray $\mathbb{R}_{\geq 0} n_i$.

Since all three branches are active, complementary slackness — the statement that a multiplier attached to a slack constraint must vanish — is vacuous here.

Corollary 10.8 (Reduction to a scalar inequality)**.** *If $\lambda_1 H_1 + \lambda_2 H_2 + \lambda_3 H_3 \geq 2$, then $D \geq 1$, contradicting $D \leq D_0 < 1$.*

Proof. Immediate from (10.2). $\square$

The rest of the book proves the inequality $\lambda_1 H_1 + \lambda_2 H_2 + \lambda_3 H_3 \geq 2$. Section 10.8 below disposes of the case in which some oriented sine is nonpositive; Chapter 11 handles the case of positive sines with a cyclic owner tournament; Chapter 12 handles positive sines with a transitive owner tournament; and Chapter 13 checks that these three cases are exhaustive and assembles the contradiction.

Remark 10.9 (Where complete contact is used). Complete contact was used twice above. Hypothesis (i) of the Farkas lemma requires only $b_i(x) > 0$, and feasibility alone gives $b_i(x) \geq H_i \geq 1$. Similarly, without contact the Euler computation would still give

$$2D = \sum_i \lambda_i b_i(x) \ \geq\ \sum_i \lambda_i H_i,$$

using $\lambda_i \geq 0$ and $b_i(x) \geq H_i$, an inequality in the useful direction, so Corollary 10.8 would survive. Complete contact is needed for the sine system of §10.5, whose derivation dots a vector identity with n_i and so requires the equalities $\langle d_i, n_i\rangle = H_i$; with an inequality the sign of the error term is uncontrolled.

Example 10.10 (The equality configuration)**.** The following configuration is used as a running check throughout the chapter. Take three axis-parallel unit squares, all frames equal to the standard frame $F = (e_1, e_2)$, with centres

$$P_1 = (0,0), \qquad P_2 = (1,0), \qquad P_3 = (0,1).$$

These squares have pairwise disjoint interiors and the centre triangle is right-angled at P_1, hence non-obtuse.

Orientation and area. $\det(P_2 - P_1, P_3 - P_1) = \det((1,0),(0,1)) = 1 > 0$, so $\sigma = +1$ and $D = 1$. For this example only we use coordinates for the quarter turn: $J_{+1}(a,b) = (-b, a)$.

Edges. $d_1 = (1,0)$, $d_2 = P_3 - P_2 = (-1,1)$, $d_3 = P_1 - P_3 = (0,-1)$, and indeed $d_1 + d_2 + d_3 = 0$.

Thresholds. All three frames coincide, so all three frame distances are 0 and

$$H_1 = H_2 = H_3 = H(0) = \tfrac{1+\cos 0+\sin 0}{2} = 1.$$

Normals and contact. Take $n_1 = (1,0)$, $n_2 = (0,1)$, $n_3 = (0,-1)$. Each is an axis of the common frame, so each of the three branches is owned, by either of its two squares. And

$$\langle d_1, n_1\rangle = 1, \qquad \langle d_2, n_2\rangle = \langle(-1,1),(0,1)\rangle = 1,$$

$$\langle d_3, n_3\rangle = \langle(0,-1),(0,-1)\rangle = 1,$$

so $\langle d_i, n_i\rangle = 1 = H_i$ for each i: complete contact holds.

Multipliers. By Lemma 10.3, $g = (-J_{+1}P_3,\ J_{+1}P_2) = (-(-1,0),\ (0,1)) = \big((1,0),(0,1)\big)$, while

$$a_1 = \big((1,0),(0,0)\big), \qquad a_2 = \big((0,-1),(0,1)\big), \qquad a_3 = \big((0,0),(0,1)\big).$$

Reading $g = \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3$ blockwise: the first block gives $(1,0) = \lambda_1(1,0) + \lambda_2(0,-1)$, so $\lambda_1 = 1$ and $\lambda_2 = 0$; the second gives $(0,1) = \lambda_2(0,1) + \lambda_3(0,1)$, so $\lambda_2 + \lambda_3 = 1$ and hence $\lambda_3 = 1$. Thus

$$(\lambda_1,\lambda_2,\lambda_3) = (1,0,1), \qquad \sum_i \lambda_i H_i = 1 + 0 + 1 = 2 = 2D.$$

Three features of this configuration recur later.

(a) *A multiplier may vanish.* Here $\lambda_2 = 0$. Nothing in the argument may assume $\lambda_i > 0$ without proof (see Exercise 10.7).

(b) *The bound is attained.* $\sum_i \lambda_i H_i = 2$, so every inequality proved in the remaining chapters must be sharp at this configuration.

(c) *Ownership is a choice.* All three frames coincide, so every branch here admits either of its squares as owner. Ownership is data recorded at the freeze, not something read off the geometry: all eight owner assignments, two of them cyclic and six transitive, are available on this one triple of squares, and the proof must work for any of them; §10.9 classifies them.

This configuration has $D = 1$, so it is not a feasible point of the Chapter 9 minimisation, which requires $D \le D_0 < 1$. It lies on the boundary of the excluded region, which is what makes it a sharp test of the inequalities below.

10.3 Resolving into plane components: the reciprocal equations

Proposition 10.11 (Reciprocal equations)**.** *At the minimiser,*

$$J_\sigma d_1 = f_2 - f_3, \qquad J_\sigma d_2 = f_3 - f_1, \qquad J_\sigma d_3 = f_1 - f_2. \tag{10.3}$$

Proof. The multiplier equation $g = \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3$ is an identity in $\mathbb{R}^4$. Its two planar blocks, using Lemma 10.3 on the left and Definition 10.2 on the right, are

$$-J_\sigma P_3 = \lambda_1 n_1 - \lambda_2 n_2 = f_1 - f_2, \qquad J_\sigma P_2 = \lambda_2 n_2 - \lambda_3 n_3 = f_2 - f_3.$$

Since $P_1 = 0$ we have $d_1 = P_2$ and $d_3 = -P_3$, so the left-hand sides are $J_\sigma d_3$ and $J_\sigma d_1$: these are the third and the first of the displayed equations. For the middle one, add all three purported right-hand sides and all three purported left-hand sides:

$$(f_2 - f_3) + (f_3 - f_1) + (f_1 - f_2) = 0, \qquad J_\sigma d_1 + J_\sigma d_2 + J_\sigma d_3 = J_\sigma(d_1 + d_2 + d_3) = 0.$$

Both sums vanish, so the middle equation follows by subtracting the two we have proved from these identities. $\square$

Both sides sum to zero: $d_1 + d_2 + d_3 = 0$ closes the centre triangle, and $\sum_i (f_{i+1} - f_{i+2}) = 0$ telescopes. Two vector equations can therefore be written as three symmetric ones, and conversely any two of the three imply the third, given $d_1 + d_2 + d_3 = 0$ (Exercise 10.1).

Example 10.12 (The reciprocal equations on the equality configuration)**.** Continue Example 10.10, where $\lambda = (1, 0, 1)$ and hence

$$f_1 = (1, 0), \qquad f_2 = (0, 0), \qquad f_3 = (0, -1).$$

Then, with $J = J_{+1}$,

$$Jd_1 = J(1, 0) = (0, 1) = f_2 - f_3, \qquad Jd_2 = J(-1, 1) = (-1, -1) = f_3 - f_1,$$

$$Jd_3 = J(0, -1) = (1, 0) = f_1 - f_2,$$

all three as claimed. This is how the multipliers were computed above: the reciprocal equations are four scalar equations of rank 3 in the three unknowns $\lambda_1, \lambda_2, \lambda_3$, and so determine them uniquely (Exercise 10.4; the criterion there is that the three normals are not all pairwise parallel, which must be checked here, since n_2 and n_3 are parallel). The sine system of §10.5 does not determine them (Example 10.18).

10.4 Maxwell's reciprocal diagram

Proposition 10.11 says that the three points f_1, f_2, f_3 are the vertices of a triangle whose side vectors are the quarter turns of the centre triangle's edges, while each vertex f_i is pinned to the ray through n_i at distance λ_i from the origin. In Maxwell's language this is a *reciprocal diagram*: the multipliers are contact forces acting along the frozen normals, and the assertion that the force polygon closes is the statement of equilibrium. The classical reference is J. C. Maxwell, "On reciprocal figures and diagrams of forces", *Philosophical Magazine* (4) **27** (1864). No physical principle is used below; f_i is a name for $\lambda_i n_i$.

Proposition 10.13 (Shape and size of the force triangle)**.** *Let f_1, f_2, f_3 be as above.*

(i) The map $y \mapsto -J_\sigma y + f_2$, a rotation through $\sigma\pi/2 + \pi$ followed by a translation, sends $P_1 \mapsto f_2$, $P_2 \mapsto f_3$, $P_3 \mapsto f_1$. In particular the triangle $f_1 f_2 f_3$ is congruent to the centre triangle.

(ii) Its doubled signed area equals D:

$$\sigma \det(f_2 - f_1,\ f_3 - f_1) = D.$$

Proof. (i) Read the side vectors off (10.3):

$$f_2 - f_1 = -J_\sigma d_3, \qquad f_3 - f_2 = -J_\sigma d_1, \qquad f_1 - f_3 = -J_\sigma d_2.$$

Write $L = -J_\sigma$. This is a linear isometry, by Proposition 2.8(iv), and indeed the rotation through $\sigma\pi/2 + \pi$, since J_σ is the rotation through $\sigma\pi/2$ (Definition 2.7) and $-I$ is the rotation through π. Since $P_1 = 0$, the affine map $\Psi(y) = Ly + f_2$ sends $P_1 \mapsto f_2$. Then $\Psi(P_2) - \Psi(P_1) = Ld_1 = f_3 - f_2$, so $\Psi(P_2) = f_3$; and $\Psi(P_3) - \Psi(P_2) = Ld_2 = f_1 - f_3$, so $\Psi(P_3) = f_1$. As Ψ is an isometry, the two triangles are congruent.

(ii) Using $f_2 - f_1 = -J_\sigma d_3$ and $f_3 - f_1 = -(f_1 - f_3) = J_\sigma d_2$, and the fact that J_σ preserves determinants (Proposition 2.8(v)),

$$\det(f_2 - f_1, f_3 - f_1) = \det(-J_\sigma d_3,\ J_\sigma d_2) = -\det(J_\sigma d_3, J_\sigma d_2)$$

$$= -\det(d_3, d_2) = \det(d_2, d_3).$$

Now $d_3 = -d_1 - d_2$, so $\det(d_2, d_3) = \det(d_2, -d_1 - d_2) = -\det(d_2, d_1) = \det(d_1, d_2)$. Multiplying by σ gives $\sigma \det(f_2 - f_1, f_3 - f_1) = \sigma \det(d_1, d_2) = D$. □

The oriented sines give a third expression for the area of the force triangle.

Definition 10.14 (Oriented sines)**.** The *oriented sines* of the frozen normals are

$$s_1 = \sigma \det(n_1, n_2), \qquad s_2 = \sigma \det(n_2, n_3), \qquad s_3 = \sigma \det(n_3, n_1).$$

Note the index convention: s_i pairs n_i with n_{i+1}. Since the n_i are unit vectors, $|\det(n_i, n_j)| \le 1$ and hence $s_i \in [-1, 1]$ for each i; and $s_i = 0$ if and only if n_i and n_{i+1} are parallel, that is $n_{i+1} = \pm n_i$.

Proposition 10.15 (Shoelace form of the area)**.** $D = \lambda_1\lambda_2 s_1 + \lambda_2\lambda_3 s_2 + \lambda_3\lambda_1 s_3$.

Proof. Expanding by bilinearity and antisymmetry of the determinant,

$$\begin{aligned}\det(f_2 - f_1, f_3 - f_1) &= \det(f_2, f_3) - \det(f_2, f_1) - \det(f_1, f_3)\\ &= \det(f_1, f_2) + \det(f_2, f_3) + \det(f_3, f_1),\end{aligned}$$

which is the shoelace decomposition of the triangle $f_1 f_2 f_3$ into the three signed triangles it makes with the origin. Multiply by σ and use Proposition 10.13(ii) on the left. On the right, $f_i = \lambda_i n_i$ gives $\sigma \det(f_i, f_{i+1}) = \lambda_i\lambda_{i+1}\, \sigma \det(n_i, n_{i+1}) = \lambda_i\lambda_{i+1} s_i$ for each i. $\square$

Proposition 10.15 has two uses. It gives a second route to the area identity (10.2), taken in Corollary 10.17 below; and it expresses the signed sub-areas of the force triangle as $\sigma \det(f_i, f_{i+1}) = \lambda_i\lambda_{i+1} s_i$, which gives the case split of §10.8 its geometric meaning. That reading is recorded in §10.8, and the statement it yields about the three rays $\mathbb{R}_{\ge 0} n_i$ is the starting point of the winding argument of Chapter 11.

On the equality configuration, $f_1 = (1, 0)$, $f_2 = (0, 0)$, $f_3 = (0, -1)$, and $\sigma \det(f_2 - f_1, f_3 - f_1) = \det((-1, 0), (-1, -1)) = 1 = D$, in agreement with Proposition 10.13(ii). The force triangle is a right isosceles triangle with legs 1, congruent to the centre triangle.

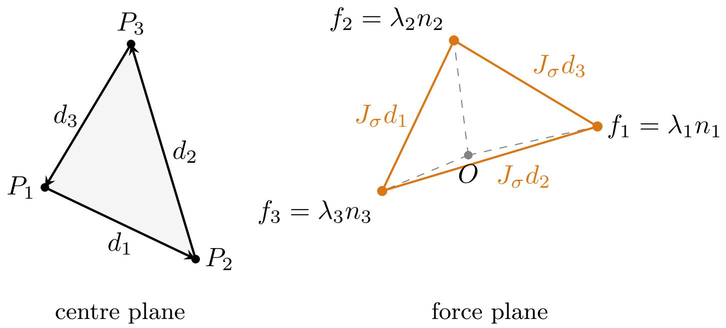


Figure 10.1: The reciprocal diagram. *Left*: the centre triangle with cyclic edges d_1, d_2, d_3. *Right*: the force plane. The points $f_i = \lambda_i n_i$ lie on the rays spanned by the frozen normals (dashed), and by (10.3) the side $f_3 f_2$ is $J_\sigma d_1$, the side $f_1 f_3$ is $J_\sigma d_2$, and the side $f_2 f_1$ is $J_\sigma d_3$. The origin O of the force plane is not P_1; the two panels live in unrelated copies of $\mathbb{R}^2$.

10.5 The sine system

The reciprocal equations are vector equations, and the case analysis of the next two chapters needs scalars. Dotting with the normals produces them.

Theorem 10.16 (The sine system)**.** *At the minimiser,*

$$H_1 = \lambda_2 s_1 + \lambda_3 s_3, \qquad H_2 = \lambda_3 s_2 + \lambda_1 s_1, \qquad H_3 = \lambda_1 s_3 + \lambda_2 s_2. \tag{10.4}$$

Proof. Apply J_σ to the first reciprocal equation $J_\sigma d_1 = f_2 - f_3$ and use $J_\sigma^2 = -I$:

$$-d_1 = J_\sigma(f_2 - f_3), \qquad \text{that is} \qquad d_1 = J_\sigma(f_3 - f_2).$$

Now dot with n_1, use $\langle J_\sigma u, v\rangle = \sigma \det(u, v)$ with $u = f_3 - f_2$ and $v = n_1$, and finally use complete contact $\langle d_1, n_1\rangle = H_1$:

$$\begin{aligned}H_1 = \langle d_1, n_1\rangle = \langle J_\sigma(f_3 - f_2), n_1\rangle &= \sigma \det(f_3 - f_2,\ n_1)\\ &= \lambda_3\, \sigma \det(n_3, n_1) - \lambda_2\, \sigma \det(n_2, n_1).\end{aligned}$$

The first term is $\lambda_3 s_3$ by definition. In the second, $\sigma \det(n_2, n_1) = -\sigma \det(n_1, n_2) = -s_1$, so the second term contributes $+\lambda_2 s_1$. Hence $H_1 = \lambda_2 s_1 + \lambda_3 s_3$.

The second and third equations are obtained in identical fashion from $J_\sigma d_2 = f_3 - f_1$ and $J_\sigma d_3 = f_1 - f_2$, dotting with n_2 and n_3 respectively; Exercise 10.1 asks for the two computations in full. □

The incidence pattern. In compact form the sine system reads

$$H_i = \lambda_{i+1} s_i + \lambda_{i+2} s_{i+2} \qquad (i = 1, 2, 3),$$

so the equation for H_i is the one equation of the system in which λ_i does not appear. Equivalently, each multiplier λ_i appears twice, in the equation for H_{i+1} (multiplied by s_i) and in the equation for H_{i+2} (multiplied by s_{i+2}); each oriented sine likewise appears twice. This pattern determines the coefficients in the weighting identity of Chapter 11, which are fixed by requiring the multipliers to cancel.

Corollary 10.17 (Consistency)**.** *Multiplying the i-th equation of* (10.4) *by λ_i and summing,*

$$\sum_{i=1}^{3} \lambda_i H_i = 2\big(\lambda_1\lambda_2 s_1 + \lambda_2\lambda_3 s_2 + \lambda_3\lambda_1 s_3\big) = 2D,$$

which is the area identity (10.2) *again.*

Proof. The three products are

$$\lambda_1 H_1 = \lambda_1\lambda_2 s_1 + \lambda_1\lambda_3 s_3, \quad \lambda_2 H_2 = \lambda_2\lambda_3 s_2 + \lambda_2\lambda_1 s_1, \quad \lambda_3 H_3 = \lambda_3\lambda_1 s_3 + \lambda_3\lambda_2 s_2,$$

and each of the three products $\lambda_1\lambda_2 s_1$, $\lambda_2\lambda_3 s_2$, $\lambda_3\lambda_1 s_3$ occurs exactly twice in the sum. The final equality is Proposition 10.15. □

This is an independent check on the algebra. The identity $\sum_i \lambda_i H_i = 2D$ has now been derived twice from the multiplier equation $g = \sum_i \lambda_i a_i$: once by Euler homogeneity in $\mathbb{R}^4$ (Theorem 10.6), and once by a shoelace computation in the force plane followed by the incidence pattern of the sine system. The two derivations have no step in common, so a sign error in §10.3–§10.5 would show up as a discrepancy between them.

Example 10.18 (The sine system on the equality configuration)**.** Continue Example 10.10, where $n_1 = (1, 0)$, $n_2 = (0, 1)$, $n_3 = (0, -1)$ and $\sigma = +1$. Then

$$s_1 = \det\big((1, 0), (0, 1)\big) = 1, \qquad s_2 = \det\big((0, 1), (0, -1)\big) = 0,$$

$$s_3 = \det\big((0, -1), (1, 0)\big) = 1.$$

With $H_1 = H_2 = H_3 = 1$ the system (10.4) reads

$$1 = \lambda_2 + \lambda_3, \qquad 1 = \lambda_1, \qquad 1 = \lambda_1,$$

This system is underdetermined: it is satisfied by $(\lambda_1, \lambda_2, \lambda_3) = (1, t, 1-t)$ for every $t \in [0, 1]$, whereas the reciprocal equations forced $t = 0$ (Example 10.12). The sine system is thus a consequence of the reciprocal equations, and is strictly weaker. It suffices here, because the target inequality holds across the solution family: every member gives $\lambda_1 + \lambda_2 + \lambda_3 = 2$ and hence $\sum_i \lambda_i H_i = 2$.

Example 10.19 (A symmetric configuration with positive sines)**.** Take $\sigma = +1$ and let n_1, n_2, n_3 be the unit vectors at angles 0, $2\pi/3$, $4\pi/3$, with n_i an axis of the frame F_i at P_i; so each square supplies exactly one normal, and each branch is owned by the earlier of its two centres in cyclic order. Then:

Sines. $s_1 = s_2 = s_3 = \sin(2\pi/3) = \sqrt{3}/2 > 0$, since consecutive normals are at directed angle $2\pi/3$.

Thresholds. The frames of a consecutive pair differ by $2\pi/3$ as directions, hence by $2\pi/3-\pi/2 = \pi/6$ as points of the frame circle $\mathbb{R}/(\pi/2)\mathbb{Z}$; and $\min\{\pi/6,\ \pi/2-\pi/6\} = \pi/6$, so each frame distance is $\pi/6$ and

$$H_1 = H_2 = H_3 = H(\pi/6) = \frac{1+\frac{\sqrt{3}}{2}+\frac{1}{2}}{2} = \frac{3+\sqrt{3}}{4} \approx 1.1830.$$

Multipliers. By symmetry set $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$. Each equation of (10.4) becomes $H = \lambda\cdot\frac{\sqrt{3}}{2} + \lambda\cdot\frac{\sqrt{3}}{2} = \lambda\sqrt{3}$, so

$$\lambda = \frac{H}{\sqrt{3}} = \frac{3+\sqrt{3}}{4\sqrt{3}} = \frac{1+\sqrt{3}}{4} \approx 0.6830.$$

Totals. $\lambda_1+\lambda_2+\lambda_3 = \frac{3}{4}(1+\sqrt{3}) \approx 2.049 > 2$, and

$$\sum_i \lambda_i H_i = 3\lambda H = 3\cdot\frac{1+\sqrt{3}}{4}\cdot\frac{3+\sqrt{3}}{4} = \frac{3(6+4\sqrt{3})}{16} = \frac{9+6\sqrt{3}}{8} \approx 2.4238,$$

so $D = \frac{1}{2}\sum_i \lambda_i H_i = (9+6\sqrt{3})/16 \approx 1.2119$. The reciprocal equations reconstruct the triangle: $d_1 = J_\sigma(f_3 - f_2) = \lambda J_{+1}(n_3 - n_2) = \lambda(\sqrt{3}, 0)$, and by symmetry the centre triangle is equilateral of side $\lambda\sqrt{3} = (3+\sqrt{3})/4 \approx 1.183$, which is acute, has all sides at least 1, and has doubled area ≈ 1.2119, in agreement.

Here the target inequality $\sum_i \lambda_i H_i \geq 2$ is strict, in contrast to Example 10.10, where it is tight. Since $D > 1$ the configuration is neither a counterexample nor a minimiser; it is a solution of the sine system lying outside the excluded region. The data also anticipate Chapter 11: the directed gaps between consecutive normals are $\gamma_1 = \gamma_2 = \gamma_3 = 2\pi/3$, summing to 2π, and their half-angle cotangents are $u = v = w = \cot(\pi/3) = 1/\sqrt{3}$, which satisfy $uv + vw + wu = 3\cdot\frac{1}{3} = 1$.

10.6 Relabelling: the S_3 action

Chapters 11 and 12 each begin with a sentence of the form "by the relabelling conventions we may assume…". Such a sentence is justified only if every symbol's transformation is known and the displayed equations of §10.3–§10.5 are shown to keep their form. This section supplies both.

Definition 10.20 (Relabelled data)**.** Let $\pi \in S_3$. The *relabelling by* π replaces the centres by $P_i' = P_{\pi(i)}$, and with them the squares and their frames, $S_i' = S_{\pi(i)}$ and $F_i' = F_{\pi(i)}$, and re-derives all data from the new vertex order:

- $d_i' = P_{i+1}' - P_i'$ and $\sigma' = \operatorname{sign}\det(P_2' - P_1',\ P_3' - P_1')$;
- the branch attached to the new pair $\{P_i', P_{i+1}'\}$ is the same frozen branch as before, with the same threshold and the same owner, the owner being a square and a square carrying its label with it. If new branch i is old branch j, then $n_i' = -n_j$ when the relabelling reverses that branch's ordered pair and $n_i' = n_j$ otherwise, with $H_i' = H_j$ in either case;
- the multipliers are relabelled with the branches, $\lambda_i' = \lambda_j$ when new branch i is old branch j.

It suffices to treat two permutations.

Lemma 10.21 (Transformation table)**.** *Let $\rho \in S_3$ be the cyclic shift with $P_i' = P_{i+1}$, and let $\tau \in S_3$ be the transposition of P_2 and P_3. Then the relabelled data are as follows.*

	σ'	d_i'	n_i'	H_i'	λ_i'	(s_1', s_2', s_3')
ρ	σ	d_{i+1}	n_{i+1}	H_{i+1}	λ_{i+1}	(s_2, s_3, s_1)
τ	$-\sigma$	$(-d_3, -d_2, -d_1)$	$(-n_3, -n_2, -n_1)$	(H_3, H_2, H_1)	$(\lambda_3, \lambda_2, \lambda_1)$	(s_2, s_1, s_3)

Here the triples in the τ-row list the values for $i = 1, 2, 3$ in order. Moreover ρ reverses no branch and τ reverses all three; ρ and τ generate S_3; and a general $\pi \in S_3$ reverses all three branches if π is odd and none if π is even, while $\sigma' = \operatorname{sign}(\pi)\,\sigma$.

Proof. The shift ρ. Here $P_1' = P_2$, $P_2' = P_3$, $P_3' = P_1$, so $d_i' = P_{i+2} - P_{i+1} = d_{i+1}$. The signed area $\det(P_2 - P_1, P_3 - P_1)$ is unchanged by a cyclic permutation of the three vertices, so $\sigma' = \sigma$. The new branch i joins the ordered pair (P_{i+1}, P_{i+2}), which is old branch $i+1$ in its original order; hence no reversal, $n_i' = n_{i+1}$, $H_i' = H_{i+1}$, $\lambda_i' = \lambda_{i+1}$. Finally

$$s_i' = \sigma' \det(n_i', n_{i+1}') = \sigma \det(n_{i+1}, n_{i+2}) = s_{i+1},$$

which is the triple (s_2, s_3, s_1).

The transposition τ. Here $P_1' = P_1$, $P_2' = P_3$, $P_3' = P_2$. Compute the edges directly:

$$d_1' = P_3 - P_1 = -d_3, \qquad d_2' = P_2 - P_3 = -d_2, \qquad d_3' = P_1 - P_2 = -d_1.$$

Also $\det(P_3 - P_1, P_2 - P_1) = -\det(P_2 - P_1, P_3 - P_1)$, so $\sigma' = -\sigma$. New branch 1 joins the ordered pair (P_1, P_3), which is old branch 3 with its order reversed, so $n_1' = -n_3$ and $H_1' = H_3$, $\lambda_1' = \lambda_3$; new branch 2 joins (P_3, P_2), old branch 2 reversed, so $n_2' = -n_2$, $H_2' = H_2$, $\lambda_2' = \lambda_2$; new branch 3 joins (P_2, P_1), old branch 1 reversed, so $n_3' = -n_1$, $H_3' = H_1$, $\lambda_3' = \lambda_1$. For the sines, the two sign flips of the normals cancel and the flip of σ survives:

$$s_1' = \sigma' \det(n_1', n_2') = -\sigma \det(-n_3, -n_2) = -\sigma \det(n_3, n_2) = \sigma \det(n_2, n_3) = s_2,$$

and in the same way $s_2' = \sigma' \det(n_2', n_3') = -\sigma \det(n_2, n_1) = \sigma \det(n_1, n_2) = s_1$ and $s_3' = \sigma' \det(n_3', n_1') = -\sigma \det(n_1, n_3) = \sigma \det(n_3, n_1) = s_3$.

Generation and parity. As permutations of $\{1, 2, 3\}$, ρ is the 3-cycle $i \mapsto i+1$ and τ is a transposition; the subgroup they generate contains an element of order 3 and an element of order 2, hence has order divisible by 6, so equals S_3. The even permutations are exactly $\mathrm{id}, \rho, \rho^2$, that is the maps $i \mapsto i + k$ modulo 3; for these, $\pi(i+1) = \pi(i) + 1$, so every ordered pair $(P_i', P_{i+1}') = (P_{\pi(i)}, P_{\pi(i)+1})$ is an old branch in its original order and no branch is reversed. The odd permutations are exactly the maps $i \mapsto c - i$ modulo 3 (there are three such c, and each such map is a transposition); for these, $\pi(i+1) = \pi(i) - 1$, so every ordered pair is an old branch reversed. Finally $\det(P_{\pi(2)} - P_{\pi(1)}, P_{\pi(3)} - P_{\pi(1)}) = \mathrm{sign}(\pi) \det(P_2 - P_1, P_3 - P_1)$, because the doubled signed area is an alternating function of the ordered vertex triple, whence $\sigma' = \mathrm{sign}(\pi)\sigma$. ☐

Proposition 10.22 (Invariance of the displayed system). *Let $\pi \in S_3$ and let the primed data be as in Definition 10.20. Then the primed data satisfy, in identical displayed form:*

$$\lambda_i' \geq 0, \qquad H_i' = H\big(d_4(F_i', F_{i+1}')\big) \geq 1, \qquad \langle d_i', n_i' \rangle = H_i',$$

each n_i' being a unit axis of the frame of the (relabelled) owner of branch i; the reciprocal equations (10.3) with $J_{\sigma'}$ and $f_i' = \lambda_i' n_i'$, the sine system (10.4), and the area identity (10.2). Moreover

$$\sum_i \lambda_i' H_i' = \sum_i \lambda_i H_i, \qquad D' = D,$$

and the condition "$s_1, s_2, s_3 > 0$" holds for the primed data if and only if it holds for the original data.

Proof. By Lemma 10.21 every $\pi \in S_3$ is a product of copies of ρ and τ, and the relabelling by a product is the composite of the relabellings; so it suffices to treat the two generators. For ρ every claim is immediate re-indexing: $\sigma' = \sigma$, no normal changes sign, and every displayed equation becomes the equation with all indices shifted by one. It remains to treat τ.

Nonnegativity and thresholds. λ_i' and H_i' are permutations of λ_i and H_i; in particular $\lambda_i' \geq 0$ and $H_i' \geq 1$. For the compatibility of the thresholds with the relabelled frames, used in Chapter 12 when it writes $A = H(d_4(F_1, F_2))$ and its companions, note that if new branch i is old branch j, then $\{F_i', F_{i+1}'\} = \{F_j, F_{j+1}\}$ as an unordered pair, so

$$H\big(d_4(F_i', F_{i+1}')\big) = H\big(d_4(F_j, F_{j+1})\big) = H_j = H_i',$$

the frame distance d_4 being symmetric in its two arguments. Finally $n_i' = \pm n_j$, and the axes of a frame come in the pairs $\pm e, \pm f$, so n_i' is again a unit axis of the frame of that branch's owner.

Contact. For each i there is j with $d'_i = -d_j$, $n'_i = -n_j$ and $H'_i = H_j$, so

$$\langle d'_i, n'_i\rangle = \langle -d_j, -n_j\rangle = \langle d_j, n_j\rangle = H_j = H'_i.$$

The relabelling flips the edge and the normal, and the two sign flips cancel.

Reciprocal equations. Since $\sigma' = -\sigma$ we have $J_{\sigma'} = -J_\sigma$ by Proposition 2.8(iii), and the primed forces are

$$f'_1 = \lambda'_1 n'_1 = \lambda_3(-n_3) = -f_3, \qquad f'_2 = -f_2, \qquad f'_3 = -f_1.$$

Now compute the three primed left- and right-hand sides:

$$J_{\sigma'} d'_1 = (-J_\sigma)(-d_3) = J_\sigma d_3 = f_1 - f_2, \qquad f'_2 - f'_3 = (-f_2) - (-f_1) = f_1 - f_2,$$

so the first primed equation holds; likewise $J_{\sigma'} d'_2 = (-J_\sigma)(-d_2) = J_\sigma d_2 = f_3 - f_1$ and $f'_3 - f'_1 = (-f_1) - (-f_3) = f_3 - f_1$; and $J_{\sigma'} d'_3 = (-J_\sigma)(-d_1) = J_\sigma d_1 = f_2 - f_3$ and $f'_1 - f'_2 = (-f_3) - (-f_2) = f_2 - f_3$. All three hold, with the sign flip of σ and the sign flips of the edges and forces cancelling each time.

Sine system. Substituting the τ-row of the table into the first primed equation,

$$\lambda'_2 s'_1 + \lambda'_3 s'_3 = \lambda_2 s_2 + \lambda_1 s_3,$$

which is the right-hand side of the third original equation, whose left-hand side is $H_3 = H'_1$. So the first primed equation is the third original one, and similarly the second primed equation is the second original one ($\lambda'_3 s'_2 + \lambda'_1 s'_1 = \lambda_1 s_1 + \lambda_3 s_2 = H_2 = H'_2$) and the third primed equation is the first original one ($\lambda'_1 s'_3 + \lambda'_2 s'_2 = \lambda_3 s_3 + \lambda_2 s_1 = H_1 = H'_3$).

Areas and totals. $\sum_i \lambda'_i H'_i = \lambda_3 H_3 + \lambda_2 H_2 + \lambda_1 H_1$, a rearrangement. And

$$D' = \sigma' \det(d'_1, d'_2) = -\sigma \det(-d_3, -d_2) = -\sigma \det(d_3, d_2)$$

$$= \sigma \det(d_2, d_3) = \sigma \det(d_1, d_2) = D,$$

the last step being the computation $\det(d_2, d_3) = \det(d_1, d_2)$ from the proof of Proposition 10.13(ii). Consequently (10.2) holds in primed form. Finally $(s'_1, s'_2, s'_3) = (s_2, s_1, s_3)$ is a permutation of the original triple, so all three are positive for the primed data exactly when all three are positive for the original data. □

Example 10.23 (Relabelling of the equality configuration by τ)**.** Apply τ to Example 10.10. The new centres are

$$P'_1 = (0,0), \qquad P'_2 = (0,1), \qquad P'_3 = (1,0),$$

so $\det(P'_2 - P'_1, P'_3 - P'_1) = \det((0,1),(1,0)) = -1$ and $\sigma' = -1$. The table gives

$$n'_1 = -n_3 = (0,1), \quad n'_2 = -n_2 = (0,-1), \quad n'_3 = -n_1 = (-1,0),$$

$$H' = (1,1,1), \quad \lambda' = (1,0,1),$$

and $s' = (s_2, s_1, s_3) = (0,1,1)$. Check two of these directly. Contact: $d'_1 = P'_2 - P'_1 = (0,1)$ and $\langle d'_1, n'_1\rangle = \langle (0,1),(0,1)\rangle = 1 = H'_1$. First sine equation: $\lambda'_2 s'_1 + \lambda'_3 s'_3 = 0\cdot 0 + 1 \cdot 1 = 1 = H'_1$. And directly from the definition, $s'_1 = \sigma' \det(n'_1, n'_2) = -\det((0,1),(0,-1)) = 0$ and $s'_2 = -\det((0,-1),(-1,0)) = -(0\cdot 0 - (-1)(-1)) = 1$, confirming the table.

Here σ changes from $+1$ to -1. No step of the argument, in this chapter or the next three, assumes $\sigma = +1$; the sign is carried symbolically so that odd relabellings remain available.

10.7 "Without loss of generality"

A "without loss of generality" has three parts:

(a) a group G acting on the data;

(b) a proof that the hypotheses and the conclusion being proved are both G-invariant;

(c) an identification of the orbits, together with the choice of one representative from each.

Part (b) is the condition that fails in incorrect appeals to "without loss of generality"; it is verified below in Proposition 10.22.

The group acts on a set of tuples of vectors and numbers defined by algebraic conditions alone, not on a set of minimisers: a relabelling permutes symbols and produces no minimisation problem for the permuted symbols to solve.

Definition 10.24 (Contact data)**.** Let $\mathcal{D}$ denote the set of tuples

$$\big(\sigma;\ P_1, P_2, P_3;\ F_1, F_2, F_3;\ n_1, n_2, n_3;\ H_1, H_2, H_3;\ \lambda_1, \lambda_2, \lambda_3;\ \text{owners}\big)$$

in which $\sigma \in \{-1, +1\}$; the P_i are points of $\mathbb{R}^2$ and the F_i frames; the owner of branch i is one of the two squares $Q(F_i, P_i)$, $Q(F_{i+1}, P_{i+1})$, and n_i is a unit axis of the owner's frame; $H_i = H(d_4(F_i, F_{i+1}))$; and, writing $d_i = P_{i+1} - P_i$, $f_i = \lambda_i n_i$ and $D = \sigma \det(d_1, d_2)$, the following hold:

$$\lambda_i \geq 0, \qquad D > 0, \qquad \langle d_i, n_i\rangle = H_i, \qquad J_\sigma d_i = f_{i+1} - f_{i+2} \qquad (i = 1, 2, 3).$$

We call such a tuple a *contact datum.* Note that $H_i \geq 1$ automatically, since $H \geq 1$ on $[0, \pi/4]$; and that the sine system (10.4) and the area identity (10.2) hold for every contact datum, since the proofs of Theorem 10.16 and Corollary 10.17 use nothing but the listed conditions.

Two conditions are absent from Definition 10.24: minimality and the normalisation $P_1 = 0$. Neither is used below. Minimality was needed only to produce the multipliers, in Theorem 10.6; and translating all three centres so that $P_1 = 0$ changes no d_i, no H_i, no λ_i and not D, by Lemma 9.13 of Chapter 9. Chapter 13 uses the same tuple under the same name and cites this definition.

Proposition 10.25 (Reduction principle)**.** *The data of the minimising configuration form a contact datum. The group S_3 acts on $\mathcal{D}$ by Definition 10.20, and the statement*

$$\lambda_1 H_1 + \lambda_2 H_2 + \lambda_3 H_3 \ \geq\ 2$$

is S_3-invariant. Consequently it suffices to prove it for one representative of each orbit.

Proof. Each condition defining $\mathcal{D}$ has been established for the minimiser above: $D > 0$ and $\langle d_i, n_i\rangle = H_i$ are the standing hypotheses, $\lambda_i \geq 0$ is Theorem 10.6, the reciprocal equations are Proposition 10.11, and the conditions on the frames, normals, owners and thresholds are the freeze.

That relabelling maps $\mathcal{D}$ to $\mathcal{D}$ is Proposition 10.22, which verifies each listed condition for the primed data; there is no minimality clause to verify. That it is a group action is the observation, used in that proof, that relabelling by a product is the composite of the relabellings, and that relabelling by the identity is the identity. Invariance of the statement is the identity $\sum_i \lambda_i' H_i' = \sum_i \lambda_i H_i$ of Proposition 10.22. The last sentence is then immediate: if the inequality holds at one point of an orbit it holds at every point of that orbit. □

Proposition 10.25 will be used to reach two normal forms:

- **Chapter 11 (cyclic):** P_i owns branch i for each i;
- **Chapter 12 (transitive):** P_1, P_2, P_3 are the source, the middle vertex and the sink of the owner tournament, in that order.

An appeal to "without loss of generality" fails when it normalises a quantity on which the conclusion depends in a way the group cannot undo. Here "assume $\sigma = +1$" would be legitimate, since odd relabellings flip σ (Lemma 10.21) and Proposition 10.22 shows the system survives them; we do not use it, as no step of the argument needs a sign. "Assume $H_1 \leq H_2 \leq H_3$" would not be legitimate: the sine system (10.4) is not symmetric in the indices, only equivariant for the S_3 action of Definition 10.20, and that action moves the sines and multipliers along with the thresholds, so an arbitrary reordering of the H_i alone is not available.

10.8 The nonpositive-sine case

One of the three cases follows from monotone estimation alone.

Proposition 10.26 (Nonpositive sine)**.** *If $s_i \le 0$ for some $i \in \{1,2,3\}$, then*

$$\lambda_1+\lambda_2+\lambda_3 \ge 2, \qquad \textit{and hence} \qquad \lambda_1 H_1+\lambda_2 H_2+\lambda_3 H_3 \ge 2.$$

Proof. Every s_j lies in $[-1,1]$, because the n_j are unit vectors; in particular $s_j \le 1$ for each j.

Suppose first that $s_3 \le 0$. In the first equation of (10.4) the term $\lambda_3 s_3$ is nonpositive, since $\lambda_3 \ge 0$; dropping it and then bounding $s_1 \le 1$,

$$H_1 = \lambda_2 s_1 + \lambda_3 s_3 \le \lambda_2 s_1 \le \lambda_2.$$

In the second equation, bound both sines by 1:

$$H_2 = \lambda_3 s_2 + \lambda_1 s_1 \le \lambda_3 + \lambda_1.$$

Since $H_1 \ge 1$ and $H_2 \ge 1$, these give $\lambda_2 \ge 1$ and $\lambda_1+\lambda_3 \ge 1$, whence $\lambda_1+\lambda_2+\lambda_3 \ge 2$. Finally, as every $\lambda_i \ge 0$ and every $H_i \ge 1$,

$$\sum_i \lambda_i H_i \;\ge\; \sum_i \lambda_i \;\ge\; 2.$$

For the remaining cases, apply the cyclic relabelling ρ^k of Lemma 10.21, which sends (s_1,s_2,s_3) to (s_2,s_3,s_1) for $k=1$ and to (s_3,s_1,s_2) for $k=2$. If $s_2 \le 0$, apply ρ^2: the new s_3' is the old s_2, and by Proposition 10.22 the relabelled data satisfy the same displayed sine system with $\lambda_i' \ge 0$ and $H_i' \ge 1$, so the case just proved applies to them and yields both $\sum_i \lambda_i' \ge 2$ and $\sum_i \lambda_i' H_i' \ge 2$. Now $\sum_i \lambda_i' H_i' = \sum_i \lambda_i H_i$, again by Proposition 10.22, and $\sum_i \lambda_i' = \sum_i \lambda_i$, the primed multipliers being a permutation of the original ones (Lemma 10.21); so both conclusions transfer. If $s_1 \le 0$, apply ρ instead, whose s_3' is the old s_1. ☐

Remark 10.27 (Sharpness). Example 10.10 has $s_2 = 0$, so it falls under Proposition 10.26, and there $\sum_i \lambda_i = \sum_i \lambda_i H_i = 2$. The proposition is therefore sharp. The extremal value is thus attained already in this case; the difficulty in Chapters 11 and 12 is that no comparably simple estimate is available there.

Standing assumption. From here to the end of Chapter 12 we assume

$$s_1 > 0, \qquad s_2 > 0, \qquad s_3 > 0.$$

The standing assumption says that the three rays $\mathbb{R}_{\ge 0} n_1$, $\mathbb{R}_{\ge 0} n_2$, $\mathbb{R}_{\ge 0} n_3$ occur in positive cyclic order for the orientation σ and so surround the origin of the force plane. By Proposition 10.15 the three signed sub-areas $\sigma \det(f_i, f_{i+1}) = \lambda_i \lambda_{i+1} s_i$ are then nonnegative; if moreover all $\lambda_i > 0$ they are strictly positive and the origin lies strictly inside the force triangle. The proviso is needed: if $\lambda_i = 0$ then f_i is the origin, which is then a vertex of the force triangle rather than an interior point (Exercise 10.6).

10.9 Tournaments on three vertices, and the owner tournament

The surviving case is organised by ownership, which is combinatorial data.

Definition 10.28 (Tournament)**.** A *tournament* on a finite vertex set is a complete graph in which each edge has been assigned a direction. The *out-degree* of a vertex is the number of edges directed away from it and its *in-degree* the number directed towards it. On three vertices, a *source* is a vertex of out-degree 2 and a *sink* a vertex of in-degree 2. Two tournaments are *isomorphic* if some bijection of their vertex sets carries directed edges to directed edges.

Lemma 10.29 (Classification)**.** *Up to isomorphism there are exactly two tournaments on three vertices: the directed* 3*-cycle, and the transitive tournament, which has a source, a middle vertex and a sink.*

Proof. There are three edges, so the three out-degrees are elements of $\{0, 1, 2\}$ summing to 3. If some vertex has out-degree 2 it is a source; the edge between the remaining two vertices then orders them, the tail being the middle vertex and the head the sink, and the tournament is transitive. Otherwise all out-degrees are at most 1 and sum to 3, so all equal 1; then all in-degrees equal 1 too. Labelling so that the edge between the first two vertices is $1 \to 2$, vertex 2 must have its unique out-edge to the remaining vertex, so $2 \to 3$, and vertex 3 must have its unique out-edge to 1, so $3 \to 1$: a directed cycle. The two are not isomorphic, since one has a vertex of out-degree 2 and the other does not, and out-degrees are preserved by isomorphism. □

Definition 10.30 (Owner tournament)**.** Direct each edge of the centre triangle away from the owner of its branch, the owner being the square fixed at the freeze, one whose frame has that branch's normal as an axis. The resulting tournament on $\{P_1, P_2, P_3\}$ is the *owner tournament* of the frozen configuration.

Corollary 10.31 (Two orbits of owner assignments)**.** *There are eight possible owner assignments, since each of the three branches may be owned by either of its two squares. Under the S_3 action of §10.6 these form exactly two orbits: the two directed* 3*-cycles, an orbit of size* 2 *with stabiliser the cyclic subgroup A_3; and the six transitive tournaments, an orbit of size* 6 *with trivial stabiliser. Hence, by Proposition 10.25, it suffices to treat one cyclic normal form and one transitive normal form.*

Proof. A relabelling permutes the vertices and carries the owner tournament to its image, so S_3 acts on the set of eight tournaments on the labelled vertex set $\{P_1, P_2, P_3\}$, and by Lemma 10.29 the two isomorphism types are unions of orbits.

Consider the directed 3-cycle $P_1 \to P_2 \to P_3 \to P_1$. The cyclic shift ρ maps it to itself, so its stabiliser contains $A_3 = \{\mathrm{id}, \rho, \rho^2\}$; and τ maps it to the reverse cycle $P_1 \to P_3 \to P_2 \to P_1$, which is a different tournament on the labelled vertex set, so the stabiliser is exactly A_3. Orbit–stabiliser gives an orbit of size $6/3 = 2$, namely the two directed 3-cycles.

Consider a transitive tournament. Its source, middle vertex and sink are determined by the tournament (they are the vertices of out-degree $2, 1, 0$), so a relabelling fixing the tournament must fix each of the three vertices and hence be the identity; the stabiliser is trivial and the orbit has size $6/1 = 6$. As $2 + 6 = 8$, these are all the tournaments and there are exactly two orbits. □

The transposition τ interchanges the two directed 3-cycles while ρ fixes each of them, so a configuration whose owner tournament is the other 3-cycle can be brought to the normal form "P_i owns branch i" only by an odd relabelling, which flips σ; this is where Proposition 10.22 for τ is needed. In the transitive case the six labelled tournaments form a single orbit, so the normal form "P_1, P_2, P_3 are the source, middle and sink in that order" is always attainable, by exactly one relabelling.

In the cyclic case each square supplies one normal, and the three normals wind once around the circle; Chapter 11 converts that winding into a single algebraic constraint on half-angle cotangents. In the transitive case the source owns two branches, so two of the normals are axes of a single frame and are therefore parallel or perpendicular; positivity of the sines excludes parallel, so they are perpendicular and one oriented sine equals 1, which reduces the problem to one parameter.

10.10 Owner arrows and branch normals

Owner arrows and branch normals are unrelated objects, and are easily confused in the arguments of the next three chapters. Figure 10.3 shows both.

The owner arrow on edge i lies along the edge P_iP_{i+1} and points from the owner of branch i to the other endpoint. Its direction records a choice made at the freeze; it is combinatorial data with no metric content.

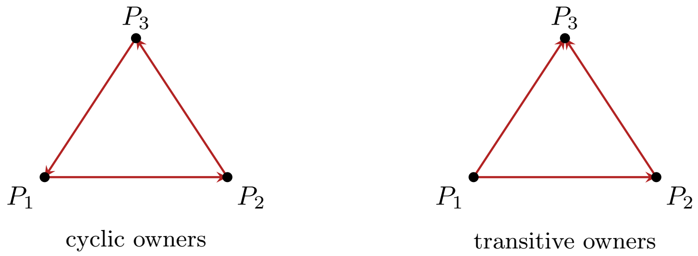


Figure 10.2: The two owner tournaments on three vertices, in the normal forms of §10.7. Each arrow points from the owner of a branch to the other endpoint of that centre-triangle edge; it is not the direction of the branch normal. On the right, P_1, P_2, P_3 are the source, the middle vertex and the sink, in that order.

The normal n_i is a unit vector, an axis of the owner's frozen frame. The only relation it bears to the edge d_i is the single scalar statement

$$\langle d_i, n_i \rangle = H_i > 0,$$

which says that n_i has a positive component along d_i. Example 10.10 illustrates this: $d_2 = (-1, 1)$ and $n_2 = (0, 1)$ make an angle of $\pi/4$, yet $\langle d_2, n_2 \rangle = 1$.

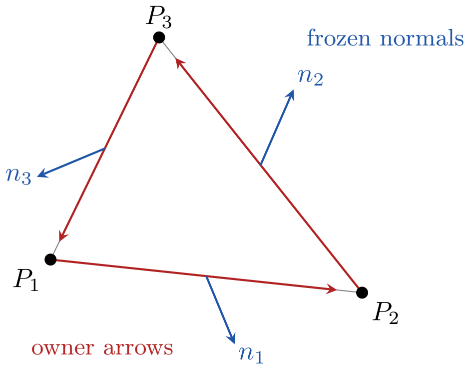


Figure 10.3: Owner arrows point from the owner of a branch to the other endpoint and lie along the edges of the centre triangle. The frozen normals are axes of frozen frames and are drawn askew, since they bear no relation to the edge directions. Every normal drawn satisfies $\langle d_i, n_i \rangle > 0$.

Exercises

Exercise 10.1. Starting from $J_\sigma d_2 = f_3 - f_1$ and $J_\sigma d_3 = f_1 - f_2$, derive the second and third equations of the sine system (10.4) in full, taking care with the sign of $\sigma \det(n_j, n_i)$ at each step. Then show that, given $d_1 + d_2 + d_3 = 0$, any two of the three reciprocal equations (10.3) imply the third.

Exercise 10.2. In Example 10.10 replace the frozen normal $n_2 = (0, 1)$ by $n_2 = (-1, 0)$, which is also an axis of both frames of that pair and also satisfies $\langle d_2, n_2 \rangle = 1 = H_2$. Recompute s_1, s_2, s_3; solve the reciprocal equations for $\lambda_1, \lambda_2, \lambda_3$; and verify the sine system and the area identity (10.2). Which oriented sine is now zero, and does the value of $\sum_i \lambda_i H_i$ change?

Exercise 10.3. Prove Proposition 10.26 directly for $i = 1$ and for $i = 2$, without any relabelling. That is: for each i, identify which equation of (10.4) yields $\lambda_{i+2} \geq 1$ and which yields $\lambda_i + \lambda_{i+1} \geq 1$. (Hint: the second is the unique equation of the system in which s_i does not appear.) Then check that Example 10.10 attains equality in the resulting bound.

Exercise 10.4. Show that a_1, a_2, a_3 are linearly dependent in $\mathbb{R}^4$ if and only if n_1, n_2, n_3 are pairwise parallel, that is, if and only if $s_1 = s_2 = s_3 = 0$. Deduce that under the standing assumption $s_1, s_2, s_3 > 0$ the multipliers $\lambda_1, \lambda_2, \lambda_3$ are uniquely determined by g. Reconcile this with Example 10.18, in which the sine system alone did not determine them.

Exercise 10.5. Assume $s_1, s_2, s_3 > 0$. Using Exercise 10.4, show that a_1, a_2, a_3 span a three-dimensional subspace of $\mathbb{R}^4$, so that its orthogonal complement is a line, and show that this line is spanned by

$$v = \big(s_2 J_\sigma n_1,\ -s_1 J_\sigma n_3\big)\ \in\ \mathbb{R}^2 \times \mathbb{R}^2.$$

(The middle computation $\langle a_2, v\rangle = 0$ needs $\langle u, J_\sigma y\rangle = \sigma \det(y, u)$.) Interpret v as a velocity: by linearity of the branch functions, all three of b_1, b_2, b_3 are constant, and not only constant to first order, along the whole line $t \mapsto x + tv$. Deduce that $\langle g, v\rangle = 0$, and then show by direct expansion that

$$D(x + tv) = D + t^2\, s_1 s_2 s_3,$$

so that the vanishing of the linear term is visible in the formula.

Exercise 10.6. Prove that $\det(y_2, y_3)\, y_1 + \det(y_3, y_1)\, y_2 + \det(y_1, y_2)\, y_3 = 0$ for any three vectors $y_1, y_2, y_3 \in \mathbb{R}^2$. (If y_1, y_2 are independent, write $y_3 = \alpha y_1 + \beta y_2$ and expand; otherwise treat the degenerate case separately.) Apply it to the forces and use Proposition 10.15 to show that the origin of the force plane has barycentric coordinates

$$\frac{1}{D}\big(\lambda_2\lambda_3 s_2,\ \lambda_3\lambda_1 s_3,\ \lambda_1\lambda_2 s_1\big)$$

with respect to the triangle $f_1 f_2 f_3$. Assuming $s_1, s_2, s_3 > 0$, deduce that the origin lies in the closed force triangle; that it lies strictly inside if and only if all three λ_i are positive; that if $\lambda_i = 0$ the origin is the vertex f_i; and that at most one λ_i can vanish. Check the statement on Example 10.10, where the sines are not all positive but the identity holds regardless.

Exercise 10.7. Suppose some multiplier vanishes. By Proposition 10.22 a cyclic relabelling ρ^k preserves the sine system, the multipliers and the thresholds; carry out this relabelling explicitly in the two cases $\lambda_2 = 0$ and $\lambda_3 = 0$, so that the vanishing multiplier may be taken to be λ_1. So let $\lambda_1 = 0$. Using the sine system, show first that $\lambda_3 > 0$ and $s_2 > 0$; then that

$$\lambda_2\lambda_3 s_2^2 = H_2 H_3 \geq 1;$$

and finally that

$$\lambda_1 H_1 + \lambda_2 H_2 + \lambda_3 H_3 = 2\lambda_2\lambda_3 s_2\ \geq\ 2.$$

Where is the bound $s_2 \leq 1$ used? Conclude that the case analysis of Chapters 11 and 12 may, if convenient, assume all three multipliers are strictly positive.

Exercise 10.8. Suppose $H_1 = H_2 = H_3 = 1$.

(a) Using the strict monotonicity of the excess $E = H - 1$ on $[0, \pi/4]$ (Proposition 5.24(ii)) and the fact that d_4 vanishes exactly on equal frame classes (Lemma 5.20(c)), show that F_1, F_2, F_3 have the same axis set, and hence that each n_i lies in $\{\pm e, \pm f\}$ for one orthonormal pair (e, f).

(b) Deduce that $s_i \in \{0, 1, -1\}$ for each i, and that $s_i = 1$ forces $n_{i+1} = J_\sigma n_i$. (For the second part, note that $s_i = \langle J_\sigma n_i, n_{i+1}\rangle$ and use the equality case of Cauchy–Schwarz.)

(c) Conclude that $s_1, s_2, s_3 > 0$ is impossible, and hence that under the standing assumption of §10.8 at least one threshold satisfies $H_i > 1$.

Finally, explain why part (c) is not enough to prove $\sum_i \lambda_i H_i \geq 2$: exhibit nonnegative λ_i and numbers $s_i \in (0, 1]$, $H_i \geq 1$ satisfying the sine system as an abstract linear system with $\sum_i \lambda_i H_i < 2$, and identify which geometric constraint on the triple (s_1, s_2, s_3) that example violates.

Chapter 11

The Cyclic Case

This chapter assumes. From Chapter 2: the quarter turn J_σ, with $J_\sigma^2 = -I$, $\langle J_\sigma u, v\rangle = \sigma \det(u, v)$ and skew-adjointness $\langle u, J_\sigma v\rangle = -\langle J_\sigma u, v\rangle$; the orthonormality of $(a, J_\sigma a)$ for a unit vector a; and the directed angle (Lemma 2.3). From Chapter 3: the cotangent half-angle formulas (Proposition 3.35), the two sign regimes (Lemma 3.36), which supply the function Φ of Section 11.5, and the angle sum as an algebraic constraint (Theorem 3.42). Also from that chapter: denominator trading (Corollary 3.43) and the cyclic dictionary built from it (Example 3.44); the fact that at most one of three positive numbers with $uv + vw + wu = 1$ exceeds 1 (Lemma 3.18); and the τ-estimate (Proposition 3.22), which combines the Engel form of Cauchy–Schwarz with vanishing numerators (Proposition 3.21) and the symmetric-function computation of Example 3.17. We use in addition $u + 1/u \geq 2$ (Corollary 3.2), the properties of Φ recorded in Example 3.37, and the peeling computations of Examples 3.10, 3.6 and 3.9. From Chapter 5: the projected half-width $w_F(n)$, with its value $\frac{1}{2}$ on an axis of F and its other-frame formula (Lemma 5.4(ii),(iii)); ownership (Definition 5.16); and the frame circle, the frame distance d_4, the owned-branch threshold $H(t) = (1 + \cos t + \sin t)/2 \geq 1$ (Lemma 5.22) and its two-normal form (Corollary 5.23). From Chapter 10: the oriented sines $s_i = \sigma \det(n_i, n_{i+1})$, the Farkas multipliers $\lambda_i \geq 0$, the sine system, the relabelling conventions, and the classification of the owner tournament into the cyclic and transitive shapes.

Standing hypotheses. Everything in this chapter is stated under the hypotheses assembled in Chapters 9 and 10 and restated here.

(H1) A counterexample to Theorem 1.1 with doubled area $D_0 < 1$ has been assumed, and one owned separating branch has been frozen for each of the three pairs. The frames F_1, F_2, F_3, the unit normals n_1, n_2, n_3, the thresholds H_1, H_2, H_3, the owners, and the orientation sign $\sigma \in \{-1, +1\}$ are constants; only the centres vary.

(H2) $x = (P_2, P_3)$ is a minimiser of the relaxed problem; the centre triangle is acute and $0 < D \leq D_0 < 1$.

(H3) All three branches are active at the minimiser: $\langle d_i, n_i\rangle = H_i$ for $i = 1, 2, 3$, where $d_i = P_{i+1} - P_i$.

(H4) There are Farkas multipliers $\lambda_1, \lambda_2, \lambda_3 \geq 0$ satisfying the sine system

$$H_1 = \lambda_2 s_1 + \lambda_3 s_3, \qquad H_2 = \lambda_3 s_2 + \lambda_1 s_1, \qquad H_3 = \lambda_1 s_3 + \lambda_2 s_2, \tag{11.1}$$

where $s_1 = \sigma \det(n_1, n_2)$, $s_2 = \sigma \det(n_2, n_3)$, $s_3 = \sigma \det(n_3, n_1)$.

(H5) $s_1, s_2, s_3 > 0$.

(H6) The owner tournament is cyclic; by the relabelling conventions of Chapter 10 we may and do assume that P_i owns branch i, the branch joining P_i to P_{i+1}, for each i. All indices are read modulo 3 in $\{1, 2, 3\}$.

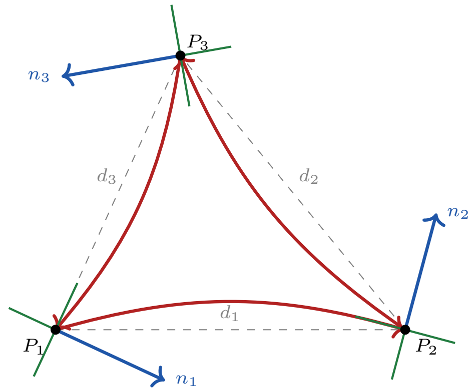


Figure 11.1: The cyclic tournament. Each red arrow points from the owner of a branch to the other endpoint; the arrows are drawn bent, away from the edges, because they are not normal directions. The blue normals n_i are axes of the frozen frames F_i (green crosses) and are drawn askew to the edges d_i, on which they do not depend.

Throughout, "active branch" means equality in the frozen linear constraint $\langle d_i, n_i\rangle = H_i$, not geometric tangency of squares. The frames and normals are frozen data; only the numbers H_i, s_i, λ_i are at issue.

Chapter 10 reduced the problem to $\lambda_1 H_1 + \lambda_2 H_2 + \lambda_3 H_3 \geq 2$ in three cases, and settled the case of a nonpositive oriented sine. This chapter treats the cyclic tournament. A winding argument converts (H5) and (H6) into the angle sum $\gamma_1 + \gamma_2 + \gamma_3 = 2\pi$; half-angle cotangents replace that relation by a symmetric quadratic equation and make the thresholds and sines rational functions; a choice of dual weights converts the sine system into an identity for $\lambda_1 + \lambda_2 + \lambda_3$; and a scalar inequality in three variables completes the proof.

11.1 The cyclic normalisation

The cyclic hypothesis (H6) says that each centre owns the branch to the next centre. One consequence of it is used below.

Lemma 11.1 (one normal per square)**.** *In the cyclic case, n_i is an axis of F_i, the frame of the owner P_i of branch i, and n_{i+1} is an axis of F_{i+1}, the frame of the other endpoint P_{i+1}. Hence the two frames relevant to branch i supply exactly the two normals n_i and n_{i+1}, and no data beyond n_i and n_{i+1} enters the threshold H_i.*

Proof. By (H1) each frozen branch is owned, so its normal is an axis of the owner's frame (Definition 5.16); by (H6) the owner of branch i is P_i, whose frame is F_i, so n_i is an axis of F_i, and likewise n_{i+1} is an axis of F_{i+1}. The endpoints of branch i are P_i and P_{i+1}, so $H_i = H(F_i, F_{i+1}; n_i) = w_{F_i}(n_i) + w_{F_{i+1}}(n_i)$. The first summand is $\frac{1}{2}$ by Lemma 5.4(ii), and the second is computed from n_i and the axis n_{i+1} of F_{i+1} by the other-frame formula Lemma 5.4(iii). $\square$

In the transitive case one frame supplies two of the three normals, and the threshold computation there involves an axis which is not one of n_1, n_2, n_3; the thresholds are then not determined by n_1, n_2, n_3 alone. See Chapter 12.

11.2 Directed angles and the winding argument

Chapter 2 attached to an ordered pair of unit vectors a directed angle in $[0, 2\pi)$. Adding three such angles requires the rotations themselves, not just their angles; we construct them from J_σ.

Definition 11.2. For $\theta \in \mathbb{R}$ let $\rho_\theta := (\cos\theta)I + (\sin\theta)J_\sigma$, a linear map $\mathbb{R}^2 \to \mathbb{R}^2$. (The sign σ is the frozen orientation sign of (H1) and is suppressed from the notation.)

Lemma 11.3. *Let a, b be unit vectors of $\mathbb{R}^2$ and $\alpha, \beta, \theta \in \mathbb{R}$. Then:*
(i) $\rho_0 = I$, $\rho_\alpha \rho_\beta = \rho_{\alpha+\beta}$, *and* $\rho_{\theta+2\pi} = \rho_\theta$*;*
(ii) $(a, J_\sigma a)$ *is an orthonormal basis of* $\mathbb{R}^2$*, and* ρ_θ *preserves inner products;*
(iii) $\langle a, \rho_\theta a\rangle = \cos\theta$ *and* $\sigma \det(a, \rho_\theta a) = \sin\theta$*;*
(iv) there is exactly one $\theta \in [0, 2\pi)$ *with* $b = \rho_\theta a$*, namely the directed angle* $\theta = \angle_\sigma(a, b)$ *of Lemma 2.3; in particular* $\rho_\theta a = a$ *if and only if* $\theta \in 2\pi\mathbb{Z}$*.*

Proof. (i) $\rho_0 = I$ is immediate. Expanding and using $J_\sigma^2 = -I$ (Proposition 2.8(i)),

$$\rho_\alpha\rho_\beta = (\cos\alpha\cos\beta)I + (\cos\alpha\sin\beta + \sin\alpha\cos\beta)J_\sigma + (\sin\alpha\sin\beta)J_\sigma^2,$$

which by $J_\sigma^2 = -I$ equals

$$(\cos\alpha\cos\beta - \sin\alpha\sin\beta)I + (\cos\alpha\sin\beta + \sin\alpha\cos\beta)J_\sigma = \rho_{\alpha+\beta}$$

by the addition formulas for cos and sin. Periodicity is periodicity of cos and sin.

(ii) $\langle a, J_\sigma a\rangle = -\langle J_\sigma a, a\rangle = -\sigma\det(a, a) = 0$ by skew-adjointness and Proposition 2.8(ii),(iii), and $\langle J_\sigma a, J_\sigma a\rangle = -\langle a, J_\sigma^2 a\rangle = \langle a, a\rangle = 1$; so $(a, J_\sigma a)$ is orthonormal, hence a basis of the two-dimensional space $\mathbb{R}^2$. For any unit vector a,

$$\|\rho_\theta a\|^2 = \cos^2\theta\|a\|^2 + 2\cos\theta\sin\theta\langle a, J_\sigma a\rangle + \sin^2\theta\|J_\sigma a\|^2 = \cos^2\theta + \sin^2\theta = 1,$$

and since ρ_θ is linear and preserves the norm of every vector (apply the computation to $z/\|z\|$ for $z \neq 0$, and note $\rho_\theta 0 = 0$), the polarisation identity $\langle y, z\rangle = \frac{1}{2}(\|y + z\|^2 - \|y\|^2 - \|z\|^2)$ shows it preserves inner products.

(iii) $\langle a, \rho_\theta a\rangle = \cos\theta\|a\|^2 + \sin\theta\langle a, J_\sigma a\rangle = \cos\theta$ by (ii). For the determinant, use $\sigma\det(a, \rho_\theta a) = \langle J_\sigma a, \rho_\theta a\rangle$ and expand:

$$\langle J_\sigma a, \rho_\theta a\rangle = \cos\theta\,\langle J_\sigma a, a\rangle + \sin\theta\,\langle J_\sigma a, J_\sigma a\rangle = \sin\theta.$$

(iv) By the directed-angle lemma (Lemma 2.3) there is exactly one $\gamma \in [0, 2\pi)$ with $\langle a, b\rangle = \cos\gamma$ and $\sigma\det(a, b) = \sin\gamma$. Expanding b in the orthonormal basis of (ii) gives $b = \langle a, b\rangle a + \langle J_\sigma a, b\rangle J_\sigma a = \cos\gamma\, a + \sin\gamma\, J_\sigma a = \rho_\gamma a$, using $\langle J_\sigma a, b\rangle = \sigma\det(a, b)$. Conversely if $b = \rho_\theta a$ with $\theta \in [0, 2\pi)$ then (iii) gives $\langle a, b\rangle = \cos\theta$ and $\sigma\det(a, b) = \sin\theta$, so $\theta = \gamma$ by the uniqueness just quoted. Taking $b = a$ gives $\cos\gamma = 1$ and $\sin\gamma = 0$, so $\gamma = 0$, and $\rho_\theta a = a$ holds exactly for $\theta \in 2\pi\mathbb{Z}$ by periodicity. □

Definition 11.4. For $i = 1, 2, 3$ let $\gamma_i \in [0, 2\pi)$ be the unique number with $n_{i+1} = \rho_{\gamma_i} n_i$, given by Lemma 11.3(iv). We call γ_i the *directed gap from n_i to n_{i+1}*. By Lemma 11.3(iii),

$$\langle n_i, n_{i+1}\rangle = \cos\gamma_i, \qquad s_i = \sigma\det(n_i, n_{i+1}) = \sin\gamma_i.$$

Proposition 11.5 (winding)**.** *Under the standing hypotheses, each γ_i lies in $(0, \pi)$ and*

$$\gamma_1 + \gamma_2 + \gamma_3 = 2\pi.$$

Proof. By Definition 11.4, γ_i is the directed angle $\angle_\sigma(n_i, n_{i+1})$, and $\sigma\det(n_i, n_{i+1}) = s_i > 0$ by (H5); so $\gamma_i \in (0, \pi)$ by the last clause of Lemma 2.3.

Applying Definition 11.4 three times and using Lemma 11.3(i),

$$n_1 = \rho_{\gamma_3} n_3 = \rho_{\gamma_3}\rho_{\gamma_2} n_2 = \rho_{\gamma_3}\rho_{\gamma_2}\rho_{\gamma_1} n_1 = \rho_{\gamma_1+\gamma_2+\gamma_3} n_1.$$

By the last clause of Lemma 11.3(iv), $\gamma_1 + \gamma_2 + \gamma_3 \in 2\pi\mathbb{Z}$. But each $\gamma_i \in (0, \pi)$, so $0 < \gamma_1 + \gamma_2 + \gamma_3 < 3\pi$, and the only element of $2\pi\mathbb{Z}$ in $(0, 3\pi)$ is 2π. □

Remark 11.6 (the range bounds). The argument uses only two features of the range $(0, \pi)$: the lower bounds $\gamma_i > 0$ rule out the sum 0, and the upper bounds $\gamma_i \le \pi$ cap the sum at $3\pi < 4\pi$. Between them they leave 2π as the only available multiple of 2π; strictness at the upper end is not needed, and in any case follows from $s_i > 0$, since sin vanishes at π. An upper bound of some kind is needed, since for gaps constrained only to $(0, 2\pi)$ the sum may be 4π, as happens when all three gaps equal $4\pi/3$ (Exercise 11.5). This is where the standing hypothesis $s_i > 0$ is used; nothing else in the chapter uses (H5) directly.

The three normals are consequently pairwise distinct and pairwise non-antipodal: the gap from n_i to n_{i+1} lies in $(0, \pi)$, hence is neither 0 nor π, and the gap from n_i to n_{i+2} is $\gamma_i + \gamma_{i+1} = 2\pi - \gamma_{i+2} \in (\pi, 2\pi)$, again neither 0 nor π. The relation $\gamma_i = 2\pi - \gamma_{i+1} - \gamma_{i+2}$ shows that the three gaps determine one another; only two of them are free.

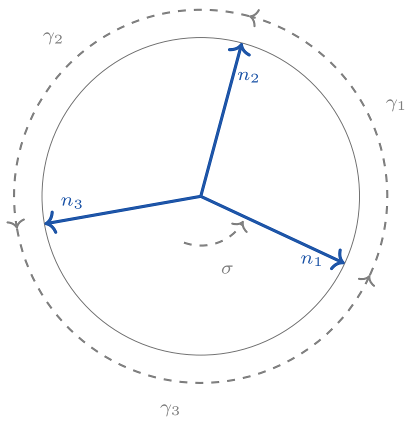


Figure 11.2: The three frozen normals on the unit circle. The inner grey arc marks the sense of increasing θ in ρ_θ, that is, the σ-orientation. Positivity of the oriented sines puts every directed gap in $(0, \pi)$, and the gaps then wind exactly once: $\gamma_1 + \gamma_2 + \gamma_3 = 2\pi$.

11.3 Half-gap cotangents and the constraint

Definition 11.7. By Proposition 11.5 each half-gap $\gamma_i/2$ lies in $(0, \pi/2)$, where cot is positive. Set

$$u := \cot(\gamma_1/2), \qquad v := \cot(\gamma_2/2), \qquad w := \cot(\gamma_3/2),$$

all three of which are strictly positive. Here and below u, v, w denote these three numbers.

Lemma 11.8 (the constraint)**.** $uv + vw + wu = 1$.

Proof. By Proposition 11.5 the three half-gaps $\gamma_i/2$ lie in $(0, \pi/2)$ and sum to π. This is the $(\Rightarrow)$ half of Theorem 3.42, applied with $x = \gamma_1/2$, $y = \gamma_2/2$, $z = \gamma_3/2$. □

Remark 11.9 (why these coordinates). In these coordinates the angle-sum relation of Proposition 11.5, a transcendental constraint on three real numbers, becomes a single symmetric quadratic equation; and each threshold H_i and each oriented sine s_i becomes a rational function of u, v, w, as the next two sections show.

The constraint is used only as a substitution: the constant 1 may be replaced by $uv + vw + wu$ wherever it occurs. It is never solved for one variable.

11.4 The sines, rationally

Corollary 11.10 (the sines, rationally)**.**

$$s_1 = \frac{2u}{(u+v)(u+w)}, \qquad s_2 = \frac{2v}{(v+w)(v+u)}, \qquad s_3 = \frac{2w}{(w+u)(w+v)}.$$

Proof. $s_i = \sin\gamma_i$ by Definition 11.4, and by Proposition 11.5 the gaps lie in $(0, \pi)$ and sum to 2π. These are the hypotheses of Example 3.44, whose three displays are the three stated formulas. □

Remark 11.11 (trading denominators for pairwise sums). The mechanism behind Corollary 11.10 is denominator trading, Corollary 3.43: under the constraint every occurrence of $1 + q^2$ may be traded for a product of two pairwise sums, and every such product may be traded back. Thus u, v, w enter the problem only through the three pairwise sums $u + v$, $v + w$, $w + u$: the trade converts a sum of three rational functions, each with its own quadratic denominator, into three fractions whose denominators are drawn from that list of three.

11.5 The cyclic factor Φ

Lemma 11.12 (cyclic thresholds)**.** *In the cyclic case,*

$$H_i = \frac{1 + |\cos\gamma_i| + \sin\gamma_i}{2} \qquad (i = 1, 2, 3),$$

and, with Φ the function

$$\Phi(q) = \frac{(1+q)\max\{1, q\}}{1+q^2} \qquad (q > 0)$$

of Lemma 3.36, we have $H_1 = \Phi(u)$, $H_2 = \Phi(v)$, $H_3 = \Phi(w)$.

Proof. Branch i is owned by P_i and n_i is an axis of F_i, so $w_{F_i}(n_i) = \frac{1}{2}$ by Lemma 5.4(ii). The other endpoint of branch i is P_{i+1}, and by Lemma 11.1 the vector $m = n_{i+1}$ is an axis of F_{i+1}; so the other-frame formula Lemma 5.4(iii) gives

$$w_{F_{i+1}}(n_i) = \frac{|\langle n_i, n_{i+1}\rangle| + |\det(n_i, n_{i+1})|}{2} = \frac{|\cos\gamma_i| + \sin\gamma_i}{2},$$

using Definition 11.4 and $|\det(n_i, n_{i+1})| = |s_i| = s_i = \sin\gamma_i > 0$. Adding the two half-widths gives $H_i = H(F_i, F_{i+1}; n_i) = (1 + |\cos\gamma_i| + \sin\gamma_i)/2$, which is the middle expression of Corollary 5.23.

Each γ_i lies in $(0, \pi)$ by Proposition 11.5, so Lemma 3.36 applies with $q = \cot(\gamma_i/2)$ and converts that expression into $\Phi(q)$. Taking $q = u, v, w$ in turn gives $H_1 = \Phi(u)$, $H_2 = \Phi(v)$, $H_3 = \Phi(w)$. $\square$

Lemma 11.13 (properties of Φ)**.** *For $q > 0$:*

(i) $\Phi(q) \geq 1$, with equality if and only if $q = 1$;

(ii) $\Phi(1/q) = \Phi(q)$;

(iii) $\max_{q>0} \Phi(q) = (1 + \sqrt{2})/2$, attained exactly at $q = \sqrt{2} + 1$ and $q = \sqrt{2} - 1$;

(iv) Φ is not differentiable at $q = 1$: its left and right derivatives there are $-\frac{1}{2}$ and $+\frac{1}{2}$.

Proof. Parts (i) and (ii) are Example 3.37(c) and (b) respectively; it remains to prove (iii) and (iv).

(iii) By (ii) it suffices to maximise on $[1, \infty)$, where $\Phi(q) = (q + q^2)/(1 + q^2)$ and

$$\Phi'(q) = \frac{(1 + 2q)(1 + q^2) - (q + q^2) \cdot 2q}{(1 + q^2)^2} = \frac{1 + 2q - q^2}{(1 + q^2)^2}.$$

The numerator vanishes on $[1, \infty)$ only at $q = 1 + \sqrt{2}$, is positive on $[1, 1 + \sqrt{2})$ and negative on $(1 + \sqrt{2}, \infty)$, so Φ increases then decreases and the maximum over $[1, \infty)$ is at $q = 1 + \sqrt{2}$. There $1 + q = 2 + \sqrt{2}$ and $1 + q^2 = 4 + 2\sqrt{2}$, so

$$\Phi(1 + \sqrt{2}) = \frac{(1 + \sqrt{2})(2 + \sqrt{2})}{4 + 2\sqrt{2}} = \frac{(1 + \sqrt{2})(2 + \sqrt{2})}{2(2 + \sqrt{2})} = \frac{1 + \sqrt{2}}{2}.$$

By (ii) the maximum over $(0, 1]$ is attained exactly at $1/(1 + \sqrt{2}) = \sqrt{2} - 1$, with the same value.

(iv) On $(0, 1]$, $\Phi(q) = (1 + q)/(1 + q^2)$ and $\Phi'(q) = \big[(1 + q^2) - (1 + q)2q\big]/(1 + q^2)^2 = (1 - 2q - q^2)/(1 + q^2)^2$, whose value at $q = 1$ is $-2/4 = -\frac{1}{2}$. On $[1, \infty)$ the derivative computed in (iii) has value $2/4 = +\frac{1}{2}$ at $q = 1$. Since Φ is continuous at $q = 1$ and given by these two rational formulas on the two sides, the one-sided derivatives at 1 are $-\frac{1}{2}$ and $+\frac{1}{2}$; they differ, so Φ is not differentiable at 1. $\square$

Remark 11.14 (consistency with Chapter 5). The function Φ is the owned-branch threshold H written in the cotangent coordinate: one has $H_i = H(t_i)$ with $t_i = d_4(F_i, F_{i+1})$ the distance from γ_i to the nearest multiple of $\pi/2$, because $|\cos\gamma| + \sin\gamma = \cos t + \sin t$ under that reduction (Corollary 5.23). Accordingly, Lemma 11.13(i) and (iii) say that the range of Φ is $[1, (1 + \sqrt{2})/2]$, which is $H([0, \pi/4])$; the minimum $q = 1$ corresponds to $\gamma = \pi/2$, i.e. perpendicular normals and $t = 0$, and the maxima $q = \sqrt{2} \pm 1$ correspond to $\gamma = \pi/4$ and $\gamma = 3\pi/4$, i.e. $t = \pi/4$.

Remark 11.15 (the corner at $q = 1$). By Lemma 11.13(iv) the corner of Φ at $q = 1$, coming from the $\max\{1, q\}$ or equivalently from the absolute value $|\cos\gamma|$, cannot be removed by rewriting. The scalar inequality of Section 11.7 therefore splits into cases according to which of u, v, w exceed 1.

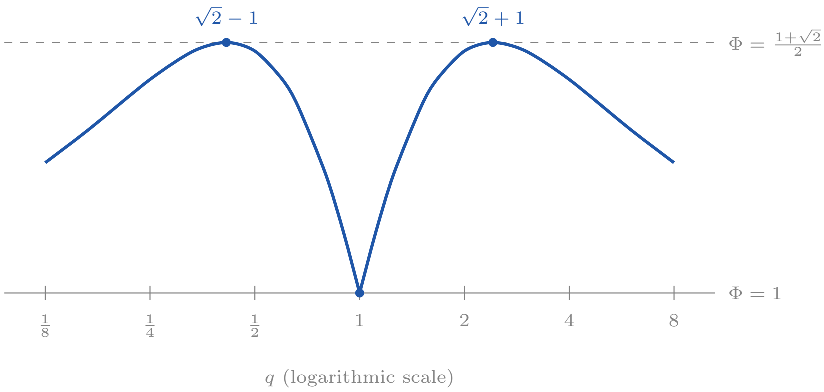


Figure 11.3: The cyclic factor $\Phi(q) = (1+q)\max\{1,q\}/(1+q^2)$, plotted against $\log q$; the vertical scale is exaggerated. The graph is symmetric under $q \mapsto 1/q$, has its two maxima $(1+\sqrt{2})/2$ at $q = \sqrt{2} \pm 1$, and has a corner at its minimum $q = 1$, where the one-sided derivatives are $\mp\frac{1}{2}$.

11.6 The weighting identity

The sine system (11.1) expresses the known thresholds in terms of the unknown multipliers. To bound $\lambda_1+\lambda_2+\lambda_3$ from below we look for weights x_1, x_2, x_3 such that the combination $x_1H_1 + x_2H_2 + x_3H_3$ has the same coefficient on each λ_i. In the cyclic case such weights exist and are unique, and can be written down explicitly.

Proposition 11.16 (weighting identity)**.** *Under the standing hypotheses,*

$$2(\lambda_1+\lambda_2+\lambda_3) = H_1(u+w) + H_2(u+v) + H_3(v+w).$$

Proof. Substitute the sine system (11.1) into the right-hand side and collect the coefficient of each multiplier.

The multiplier λ_1 occurs in the H_2 equation with sine s_1 and in the H_3 equation with sine s_3; the corresponding weights are $u+v$ and $v+w$. Its total coefficient is therefore, by Corollary 11.10,

$$s_1(u+v) + s_3(v+w) = \frac{2u(u+v)}{(u+v)(u+w)} + \frac{2w(v+w)}{(w+u)(w+v)} = \frac{2u}{u+w} + \frac{2w}{w+u} = 2.$$

The multiplier λ_2 occurs in the H_1 equation with sine s_1, weight $u+w$, and in the H_3 equation with sine s_2, weight $v+w$; its coefficient is

$$s_1(u+w) + s_2(v+w) = \frac{2u(u+w)}{(u+v)(u+w)} + \frac{2v(v+w)}{(v+w)(v+u)} = \frac{2u}{u+v} + \frac{2v}{v+u} = 2.$$

The multiplier λ_3 occurs in the H_1 equation with sine s_3, weight $u+w$, and in the H_2 equation with sine s_2, weight $u+v$; its coefficient is

$$s_3(u+w) + s_2(u+v) = \frac{2w(u+w)}{(w+u)(w+v)} + \frac{2v(u+v)}{(v+w)(v+u)} = \frac{2w}{w+v} + \frac{2v}{v+w} = 2.$$

Summing, the right-hand side equals $2\lambda_1 + 2\lambda_2 + 2\lambda_3$. $\square$

How the weights are found

Write the sine system as $H = M\lambda$ with

$$H = \begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix}, \qquad \lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & s_1 & s_3 \\ s_1 & 0 & s_2 \\ s_3 & s_2 & 0 \end{pmatrix},$$

which is (11.1) read row by row. The matrix M is symmetric, so for any weight vector $x \in \mathbb{R}^3$,

$$\langle x, H\rangle = \langle x, M\lambda\rangle = \langle M^{\mathsf{T}} x, \lambda\rangle = \langle Mx, \lambda\rangle.$$

Hence the weight vectors that give every multiplier the same coefficient 2 are the solutions of the linear system

$$Mx = (2,2,2)^{\mathsf{T}}.$$

Expanding along the first row,

$$\det M = 0\cdot(0 - s_2^2) - s_1(s_1\cdot 0 - s_2 s_3) + s_3(s_1 s_2 - 0) = 2s_1 s_2 s_3 > 0$$

by (H5), so the solution is unique: the pairing of H_1 with $u+w$, of H_2 with $u+v$ and of H_3 with $v+w$ is forced, and no other pairing gives the identity (Exercise 11.3). The solution can be found by inspection rather than by elimination. The multiplier λ_1 appears alongside s_1 and s_3, whose denominators in Corollary 11.10 are $(u+v)(u+w)$ and $(w+u)(w+v)$; these share the factor $u+w$. Weighting the H_2 equation by $u+v$ and the H_3 equation by $v+w$ cancels everything except the shared factor, leaving $2u/(u+w) + 2w/(u+w) = 2$. The remaining two coefficients are then also 2, as the proof above shows.

The same device, dual weights chosen so that the unknowns cancel, underlies the certificate construction of Chapter 12; here the linear system has a unique solution, so no free parameters remain.

11.7 The cyclic scalar inequality

It remains to prove one inequality about three positive numbers.

Theorem 11.17 (cyclic scalar inequality)**.** *Let $u, v, w > 0$ satisfy $uv + vw + wu = 1$, and let $\Phi(q) = (1+q)\max\{1,q\}/(1+q^2)$. Then*

$$R := \Phi(u)(u+w) + \Phi(v)(u+v) + \Phi(w)(v+w) \ \geq\ 4.$$

The pairing is part of the statement: $\Phi(u)$ is multiplied by $u+w$, $\Phi(v)$ by $u+v$, and $\Phi(w)$ by $v+w$.

Lemma 11.18 (cyclic invariance)**.** *The constraint $uv+vw+wu = 1$ and the quantity R are both unchanged by the relabelling $(u,v,w) \mapsto (v,w,u)$.*

Proof. The constraint is symmetric in all three letters, so it is invariant. For R, substitute: the relabelled expression is

$$\Phi(v)(v+u) + \Phi(w)(v+w) + \Phi(u)(w+u),$$

whose three terms are, respectively, the second, third and first terms of R. So the relabelled sum has the same three summands as R and hence the same value. □

Lemma 11.18 justifies the "without loss of generality" in Step 3 below, where the relabelling used is cyclic.

Lemma 11.19 (reduction to two cases)**.** *Let $u, v, w > 0$ satisfy $uv+vw+wu = 1$. Then*

$$R = \frac{(1+u)\max\{1,u\}}{u+v} + \frac{(1+v)\max\{1,v\}}{v+w} + \frac{(1+w)\max\{1,w\}}{w+u}, \tag{11.2}$$

with all three denominators positive; and in order to prove $R \geq 4$ for all such u, v, w it suffices to prove it in the two cases

- *Case 1: $0 < u, v, w \leq 1$;*
- *Case 2: $u > 1$ and $0 < v, w < 1$.*

Proof. Step 1: trade the denominators for pairwise sums. By Corollary 3.43, $1+u^2 = (u+v)(u+w)$, so

$$\Phi(u)(u+w) = \frac{(1+u)\max\{1,u\}}{(u+v)(u+w)}\,(u+w) = \frac{(1+u)\max\{1,u\}}{u+v},$$

and cyclically, using $1+v^2 = (v+w)(v+u)$ and $1+w^2 = (w+u)(w+v)$,

$$\Phi(v)(u+v) = \frac{(1+v)\max\{1,v\}}{v+w}, \qquad \Phi(w)(v+w) = \frac{(1+w)\max\{1,w\}}{w+u}.$$

Adding the three gives (11.2); the denominators $u+v$, $v+w$, $w+u$ are positive because u,v,w are.

Step 2: at most one of u,v,w exceeds 1. This is Lemma 3.18.

Step 3: the case split. By Step 2, either $u,v,w \le 1$, or exactly one of them exceeds 1. In the second situation, Lemma 11.18 lets us cyclically relabel so that the large one is u; then $u > 1$, and $v < 1$ and $w < 1$ strictly, since $v \ge 1$ together with $u > 1$ would put two variables at ≥ 1 with at least one > 1, whence $uv \ge u > 1$, contradicting the constraint as in Step 2. So the two displayed cases are exhaustive. □

11.8 Case 1: all three variables at most 1

Throughout this section $0 < u,v,w \le 1$ and $uv+vw+wu = 1$. Every $\max\{1,q\}$ in (11.2) equals 1, so

$$R = \frac{1+u}{u+v} + \frac{1+v}{v+w} + \frac{1+w}{w+u}.$$

Introduce the *deficits*

$$U := 1-u, \qquad V := 1-v, \qquad W := 1-w,$$

all lying in $[0,1)$, and put $\tau := U+V+W$.

Lemma 11.20 (excess form)**.** $R - 3 = \dfrac{V}{u+v} + \dfrac{W}{v+w} + \dfrac{U}{w+u}$.

Proof. $\dfrac{1+u}{u+v} - 1 = \dfrac{1+u-u-v}{u+v} = \dfrac{1-v}{u+v} = \dfrac{V}{u+v}$, and cyclically $\dfrac{1+v}{v+w} - 1 = \dfrac{W}{v+w}$ and $\dfrac{1+w}{w+u} - 1 = \dfrac{U}{w+u}$. Add. □

The indices are shifted: the term with denominator $u+v$ has numerator V, not U.

Proposition 11.21 (Case 1)**.** *If $0 < u,v,w \le 1$ and $uv+vw+wu = 1$, then $R \ge 4$.*

Proof. The hypotheses of the τ-estimate, Proposition 3.22, are $u,v,w \in (0,1]$ and $uv+vw+wu = 1$, and its left-hand side is the right-hand side of Lemma 11.20. Hence

$$R - 3 = \frac{V}{u+v} + \frac{W}{v+w} + \frac{U}{w+u} \;\ge\; \frac{\tau^2}{4\tau - \tau^2 - 2} \;\ge\; 1, \tag{11.3}$$

that is, $R \ge 4$. □

Case 1 thus reduces to the identification of U,V,W made in Lemma 11.20. In Proposition 3.22 each fraction X/c with $X > 0$ is rewritten as $X^2/(Xc)$ and the Engel form is applied, which is legitimate even when some X vanish (Proposition 3.21); the resulting denominator $V(u+v) + W(v+w) + U(w+u)$ reduces under the constraint to $4\tau - \tau^2 - 2$ (Example 3.17); and that expression is compared with τ^2 using $\tau^2 - (4\tau - \tau^2 - 2) = 2(\tau-1)^2 \ge 0$.

Example 11.22 (the symmetric configuration)**.** Take $\gamma_1 = \gamma_2 = \gamma_3 = 2\pi/3$, so $u = v = w = \cot(\pi/3) = 1/\sqrt{3} \approx 0.57735$; the constraint reads $3 \cdot \frac{1}{3} = 1$. Here every $\max\{1,q\} = 1$, and

$$R = 3 \cdot \frac{1 + 1/\sqrt{3}}{2/\sqrt{3}} = \frac{3(\sqrt{3}+1)}{2} = \frac{3+3\sqrt{3}}{2} \approx 4.09808 > 4.$$

In the deficit variables, $U = V = W = 1 - 1/\sqrt{3} \approx 0.42265$ and $\tau \approx 1.26795$, so

$$\frac{\tau^2}{4\tau - \tau^2 - 2} \approx \frac{1.60770}{1.46410} \approx 1.09808,$$

which is $R - 3$. The first inequality of (11.3), the Engel step, is therefore an equality here, as it must be, since the ratios $V/\big(V(u+v)\big) = 1/(u+v)$ and their two companions coincide when $u = v = w$. All the slack is in the final step $2(\tau - 1)^2 \geq 0$, which loses about 0.098.

11.9 Case 2: one variable exceeds 1

Throughout this section $u > 1$, $0 < v, w < 1$ and $uv + vw + wu = 1$. Now $\max\{1, u\} = u$ while $\max\{1, v\} = \max\{1, w\} = 1$, so (11.2) reads

$$R = \frac{u(1+u)}{u+v} + \frac{1+v}{v+w} + \frac{1+w}{w+u}.$$

Each term is bounded separately, using estimates proved in Chapter 3 under the hypotheses in force here:

$$\frac{u(1+u)}{u+v} \geq u + \frac{1-v}{1+v} \qquad \text{(Example 3.10(a))}, \tag{11.4}$$

$$\frac{1+v}{v+w} \geq \frac{(1+v)^2}{1+v^2} \qquad \text{(Example 3.6)}, \tag{11.5}$$

$$\frac{1+w}{w+u} \geq \frac{1}{u} \qquad \text{(Example 3.10(b))}. \tag{11.6}$$

Only (11.5) uses the constraint, in the form $u(v+w) + vw = 1$, which bounds $v + w$ above by $(1+v^2)/(1+v)$; since $v + w$ is a denominator, this bounds the term below. It remains to combine the three bounds.

Proposition 11.23 (Case 2)**.** *If $u > 1$, $0 < v, w < 1$ and $uv + vw + wu = 1$, then $R \geq 4$.*

Proof. Applying (11.4), (11.5) and (11.6) to the three terms in turn,

$$R \geq \left[u + \frac{1-v}{1+v}\right] + \frac{(1+v)^2}{1+v^2} + \frac{1}{u}.$$

By Example 3.9, whose hypothesis $v \in [0, 1)$ holds here, the two middle contributions satisfy $\dfrac{1-v}{1+v} + \dfrac{(1+v)^2}{1+v^2} \geq 2$, so

$$R \geq u + \frac{1}{u} + 2 \geq 2 + 2 = 4,$$

the last step by $u + 1/u \geq 2$ for $u > 0$ (Corollary 3.2). □

Proof of Theorem 11.17. By Lemma 11.19 it suffices to prove $R \geq 4$ in Cases 1 and 2, and these are Propositions 11.21 and 11.23. □

Example 11.24 (an asymmetric configuration, in Case 2)**.** Take the gaps

$$(\gamma_1, \gamma_2, \gamma_3) = (4\pi/9,\ 2\pi/3,\ 8\pi/9),$$

that is 80°, 120° and 160°; they sum to 2π, as Proposition 11.5 requires. Then

$$u = \cot 40^\circ \approx 1.19175, \qquad v = \cot 60^\circ = \tfrac{1}{\sqrt{3}} \approx 0.57735,$$

$$w = \cot 80^\circ \approx 0.17633,$$

and $uv + vw + wu = 1$ to the displayed accuracy. Only u exceeds 1, so this is Case 2 with no relabelling needed. The thresholds are

$$H_1 = \Phi(u) \approx 1.07923, \qquad H_2 = \Phi(v) \approx 1.18301, \qquad H_3 = \Phi(w) \approx 1.14086,$$

each of which may be checked against $H_i = (1 + |\cos\gamma_i| + \sin\gamma_i)/2$; for instance

$$H_3 = \frac{1 + |\cos 160^\circ| + \sin 160^\circ}{2} = \frac{1 + 0.93969 + 0.34202}{2} \approx 1.14086.$$

The three oriented sines are $s_i = \sin\gamma_i \approx (0.98481, 0.86603, 0.34202)$, and solving the sine system (11.1) gives

$$\lambda_1 \approx 0.88102, \qquad \lambda_2 \approx 0.96940, \qquad \lambda_3 \approx 0.36416,$$

all positive, with $\lambda_1 + \lambda_2 + \lambda_3 \approx 2.21459 \geq 2$. The weighting identity of Proposition 11.16 holds:

$$\begin{aligned} H_1(u+w) + H_2(u+v) + H_3(v+w) &\approx 1.47646 + 2.09287 + 0.85984 \\ &\approx 4.42918 = 2 \times 2.21459. \end{aligned}$$

Finally $2D = \sum \lambda_i H_i \approx 2.51310$, so $D \approx 1.25655 \geq 1$.

Remark 11.25 (sharpness of the constant 4). The constant 4 in Theorem 11.17 cannot be improved. For $\varepsilon \in (0,1)$ put

$$v = \varepsilon, \qquad u = w = \sqrt{1+\varepsilon^2} - \varepsilon.$$

Then $u = w > 0$, and $uv + vw + wu = 2\varepsilon u + u^2 = (u+\varepsilon)^2 - \varepsilon^2 = 1$, so the constraint holds. As $\varepsilon \to 0^+$ we have $u = w \to 1$ and $v \to 0$, whence $\Phi(u), \Phi(w) \to 1$ and $\Phi(v) \to 1$, while $u + w \to 2$, $u + v \to 1$ and $v + w \to 1$; so $R \to 1 \cdot 2 + 1 \cdot 1 + 1 \cdot 1 = 4$. Numerically $R \approx 4.00050$ at $\varepsilon = 10^{-3}$ and $R \approx 4.0000500$ at $\varepsilon = 10^{-4}$.

The limit point is $(u, v, w) = (1, 0, 1)$. It corresponds to the gap triple

$$(\gamma_1, \gamma_2, \gamma_3) = (\pi/2,\ \pi,\ \pi/2),$$

that is, to $s_2 = \sin\gamma_2 = 0$; so it sits on the boundary of the standing hypothesis (H5), where the case of a nonpositive oriented sine takes over (Chapter 10). The value 4 is approached but not attained: $R > 4$ at every admissible (u, v, w), by Exercise 11.7.

11.10 Closing the cyclic case

Proposition 11.26 (the cyclic case)**.** *Under the standing hypotheses (H1)–(H6),*

$$\lambda_1 + \lambda_2 + \lambda_3 \ \geq\ 2, \qquad \textit{and hence} \qquad \lambda_1 H_1 + \lambda_2 H_2 + \lambda_3 H_3 \ \geq\ 2.$$

Proof. By the weighting identity (Proposition 11.16) and the threshold formula (Lemma 11.12),

$$\begin{aligned} 2(\lambda_1 + \lambda_2 + \lambda_3) &= H_1(u+w) + H_2(u+v) + H_3(v+w) \\ &= \Phi(u)(u+w) + \Phi(v)(u+v) + \Phi(w)(v+w). \end{aligned}$$

By Definition 11.7 and Lemma 11.8 the numbers u, v, w are positive and satisfy $uv + vw + wu = 1$, so Theorem 11.17 bounds the right-hand side below by 4. Dividing by 2 gives $\lambda_1 + \lambda_2 + \lambda_3 \geq 2$.

For the second inequality, every $\lambda_i \geq 0$ by (H4) and every $H_i \geq 1$ (either by Lemma 5.22, or by Lemma 11.13(i) together with Lemma 11.12), so $\lambda_i H_i \geq \lambda_i$ for each i and $\lambda_1 H_1 + \lambda_2 H_2 + \lambda_3 H_3 \geq \lambda_1 + \lambda_2 + \lambda_3 \geq 2$. ☐

Chapter 10 supplies the identity $2D = \lambda_1 H_1 + \lambda_2 H_2 + \lambda_3 H_3$, so Proposition 11.26 says $D \geq 1$ at the minimiser whenever the owner tournament is cyclic. The contradiction with (H2), which says $D < 1$, is drawn in Chapter 13, once the three cases have been shown to be exhaustive.

The first inequality of Proposition 11.26 is sharp by Remark 11.25; only the second is lossy, replacing each $H_i \in [1, (1+\sqrt{2})/2]$ by 1.

In the transitive case, treated in Chapter 12, the tournament has a source owning two branches, so one frame supplies two of the three normals; those two normals are then axes of a single frame and hence perpendicular, and no winding argument is available or needed. The configuration then has one free parameter instead of the two left here by Proposition 11.5, but the three thresholds are no longer determined by the normals: they must be controlled by an inequality, $A + 1 \leq B + C$, in place of the identity $H_i = \Phi(\cdot)$ of Lemma 11.12.

Example 11.27 (a complete configuration)**.** We illustrate the chapter on one explicit configuration, extending the normals and frames of Example 10.19 by a choice of centres. Take $\sigma = +1$ and let F_1, F_2, F_3 be the frames whose axis directions are 0, $\pi/6$ and $\pi/3$. Put

$$n_1 = (1, 0), \qquad n_2 = \left(-\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}\right), \qquad n_3 = \left(-\tfrac{1}{2}, -\tfrac{\sqrt{3}}{2}\right),$$

the unit vectors at angles 0, $2\pi/3$ and $4\pi/3$. Each n_i is an axis of F_i: the angles are 0, $\pi/6 + \pi/2$ and $\pi/3 + \pi$ respectively. Let the centres be the vertices of an equilateral triangle of side $L = (3 + \sqrt{3})/4 \approx 1.18301$:

$$P_1 = (0, 0), \qquad P_2 = (L, 0), \qquad P_3 = \left(\tfrac{L}{2}, \tfrac{\sqrt{3}\,L}{2}\right).$$

Gaps and coordinates. $\gamma_1 = \gamma_2 = \gamma_3 = 2\pi/3$, so $u = v = w = 1/\sqrt{3}$ and $uv + vw + wu = 1$; this is the symmetric point of Example 11.22, and all three sines are $s_i = \sin(2\pi/3) = \sqrt{3}/2 > 0$, so (H5) holds and $\gamma_1 + \gamma_2 + \gamma_3 = 2\pi$.

Thresholds. $d_4(F_i, F_{i+1}) = \pi/6$ for each i (the frame classes are $0, \pi/6, \pi/3$ on the circle $\mathbb{R}/(\pi/2)\mathbb{Z}$, and $\pi/3$ is at distance $\pi/6$ from 0 there), so

$$H_i = H(\pi/6) = \frac{1 + \cos(\pi/6) + \sin(\pi/6)}{2} = \frac{1 + \frac{\sqrt{3}}{2} + \frac{1}{2}}{2} = \frac{3 + \sqrt{3}}{4} \approx 1.18301,$$

in agreement with $H_i = \Phi(1/\sqrt{3}) = \left(1 + \frac{1}{\sqrt{3}}\right)/\left(1 + \frac{1}{3}\right) = \frac{3}{4}\left(1 + \frac{1}{\sqrt{3}}\right) = (3 + \sqrt{3})/4$.

Complete contact. $d_1 = P_2 - P_1 = (L, 0)$, so $\langle d_1, n_1\rangle = L = H_1$. Next $d_2 = P_3 - P_2 = (-L/2, \sqrt{3}L/2)$, so $\langle d_2, n_2\rangle = \frac{L}{4} + \frac{3L}{4} = L = H_2$; and $d_3 = P_1 - P_3 = (-L/2, -\sqrt{3}L/2)$ gives $\langle d_3, n_3\rangle = \frac{L}{4} + \frac{3L}{4} = L = H_3$. All three branches are active.

Multipliers. By symmetry the sine system (11.1) reads $H = \lambda(s_1 + s_3) = \lambda\sqrt{3}$ for a common value λ, so

$$\lambda_1 = \lambda_2 = \lambda_3 = \frac{3 + \sqrt{3}}{4\sqrt{3}} = \frac{1 + \sqrt{3}}{4} \approx 0.68301 \ \geq 0.$$

Weighting identity and area. Both sides of Proposition 11.16 equal $(3 + 3\sqrt{3})/2 \approx 4.09808$: on the left $2 \cdot 3 \cdot \frac{1+\sqrt{3}}{4}$, on the right $3 \cdot \frac{3+\sqrt{3}}{4} \cdot \frac{2}{\sqrt{3}}$. Hence $\lambda_1 + \lambda_2 + \lambda_3 \approx 2.04904 \geq 2$, in accordance with Proposition 11.26, and

$$2D = \sum_i \lambda_i H_i = 3 \cdot \frac{1 + \sqrt{3}}{4} \cdot \frac{3 + \sqrt{3}}{4} = \frac{9 + 6\sqrt{3}}{8}, \qquad D = \frac{9 + 6\sqrt{3}}{16} \approx 1.21202,$$

so the centre triangle has area $D/2 \approx 0.60601 \geq \frac{1}{2}$. One checks D directly: the triangle is equilateral of side L, with doubled area $(\sqrt{3}/2)L^2 \approx 1.21202$.

The three unit squares $Q(F_i, P_i)$ have pairwise disjoint interiors, because each active branch is a separating branch (Theorem 5.9). Note that a configuration meeting all the frozen constraints with complete contact need not be near a counterexample: here $D \approx 1.21202 > 1$. Proposition 11.26 bounds D below at a hypothetical minimiser; it does not describe one.

Exercises

Exercise 11.1. Take the gap triple $(\gamma_1, \gamma_2, \gamma_3) = (\pi/2, 3\pi/4, 3\pi/4)$. Verify that the gaps sum to 2π; compute u, v, w exactly and check $uv + vw + wu = 1$; compute H_1, H_2, H_3 exactly, using both formulas of Lemma 11.12; solve the sine system (11.1) for $\lambda_1, \lambda_2, \lambda_3$; verify the weighting identity of Proposition 11.16; and check that $\lambda_1 + \lambda_2 + \lambda_3 \geq 2$ with strict inequality. Finally, using the identity $2D = \lambda_1 H_1 + \lambda_2 H_2 + \lambda_3 H_3$ of Chapter 10, compute the doubled area D of a minimiser carrying this data and check that $D \geq 1$.

Exercise 11.2. This exercise supplies the half of the correspondence between gap triples and coordinate triples that the chapter uses but does not prove. Let $u, v, w > 0$ satisfy $uv+vw+wu = 1$

and put $\gamma_1 = 2\operatorname{arccot} u$, and similarly γ_2, γ_3 from v, w, so that each $\gamma_i \in (0, \pi)$. Using the ($\Leftarrow$) half of Theorem 3.42, show $\gamma_1 + \gamma_2 + \gamma_3 = 2\pi$. Then, for any unit vector n_1, put $n_2 = \rho_{\gamma_1} n_1$ and $n_3 = \rho_{\gamma_1+\gamma_2} n_1$ and check that the three directed gaps of Definition 11.4 are exactly $\gamma_1, \gamma_2, \gamma_3$ and that $s_1, s_2, s_3 > 0$. Conclude that the coordinates of Definition 11.7 range over exactly the set $\{(u, v, w) : u, v, w > 0,\ uv + vw + wu = 1\}$, so that the algebraic form of Theorem 11.17 is equivalent to the geometric one.

Exercise 11.3. With the gaps $(4\pi/9, 2\pi/3, 8\pi/9)$ of Example 11.24, compute both

$$H_1(u + w) + H_2(u + v) + H_3(v + w) \qquad \text{and} \qquad H_1(u + v) + H_2(v + w) + H_3(w + u),$$

and show that they differ. Then prove the general statement: writing the sine system as $H = M\lambda$ as in Section 11.6, show that M is symmetric with $\det M = 2s_1s_2s_3 > 0$, deduce that the weight vector x solving $Mx = (2, 2, 2)^\mathsf{T}$ is unique, and verify that $x = (u + w,\ u + v,\ v + w)$.

Exercise 11.4. The weighting identity gives $\lambda_1 + \lambda_2 + \lambda_3$ without the individual multipliers; this exercise computes them. With M as in Section 11.6, invert M and show that

$$\lambda_1 = \frac{-s_2H_1 + s_3H_2 + s_1H_3}{2s_1s_3}, \qquad \lambda_2 = \frac{s_2H_1 - s_3H_2 + s_1H_3}{2s_1s_2}, \qquad \lambda_3 = \frac{s_2H_1 + s_3H_2 - s_1H_3}{2s_2s_3}.$$

Check the three formulas against the numbers of Example 11.24, and verify directly that they sum to $R/2$ with R as in Theorem 11.17. Note that the formulas do not exhibit $\lambda_i \geq 0$, and identify the source of the nonnegativity.

Exercise 11.5. Suppose instead that $s_1, s_2, s_3 < 0$ in the cyclic case. Show that each directed gap γ_i of Definition 11.4 lies in $(\pi, 2\pi)$ and that $\gamma_1 + \gamma_2 + \gamma_3 = 4\pi$; give an explicit triple of unit normals realising this. Then show, using the transformation table (Lemma 10.21), that the sign pattern of (s_1, s_2, s_3) is unchanged by every relabelling of Definition 10.20, so that such a configuration cannot be converted into one with positive sines; and identify which case of Chapter 10 disposes of it.

Exercise 11.6. For $\varepsilon \in (0, 1)$ set $v = \varepsilon$ and $u = w = \sqrt{1 + \varepsilon^2} - \varepsilon$. Show that $u, v, w > 0$ and $uv + vw + wu = 1$, compute

$$R(\varepsilon) = \Phi(u)(u + w) + \Phi(v)(u + v) + \Phi(w)(v + w)$$

explicitly as a function of ε, and show $R(\varepsilon) \to 4$ as $\varepsilon \to 0^+$, so that the constant 4 in Theorem 11.17 is optimal. Identify the limiting gap triple $(\gamma_1, \gamma_2, \gamma_3)$, explain why it cannot occur under the standing hypotheses of this chapter, and say which case of Chapter 10 covers configurations near it.

Exercise 11.7. The infimum 4 of Theorem 11.17 is not attained. Prove this, as follows. (a) In Case 2, show that the last step of Proposition 11.23 is strict, and conclude $R > 4$ there. (b) In Case 1, show that $R = 4$ forces equality in the second inequality of (11.3), and use the identity $\tau^2 - (4\tau - \tau^2 - 2) = 2(\tau - 1)^2$ of Example 3.17(i) to determine the only possible value of τ. (c) Show that this value of τ is incompatible with $u, v, w > 0$. (Use Lemma 3.15 to evaluate $UV + VW + WU$, and remember that $U, V, W \geq 0$ in Case 1.) (d) Deduce that $R > 4$ at every (u, v, w) with $u, v, w > 0$ and $uv + vw + wu = 1$, and hence that $\lambda_1 + \lambda_2 + \lambda_3 > 2$ and $D > 1$ strictly under the standing hypotheses (H1)–(H6). Why does this not improve Proposition 11.26?

Chapter 12

The Transitive Case

What this chapter assumes

- From Chapter 2: the quarter turn J_σ and the identities $J_\sigma^2 = -I$ and $\langle J_\sigma u, v\rangle = \sigma \det(u, v)$. Bilinearity and alternation of det. The fact that in $\mathbb{R}^2$ the unit vectors orthogonal to a given unit vector n are $\pm J_\sigma n$.
- From Chapter 3: the half-angle substitution $r = \tan(t/2)$ with

$$\cos t = \frac{1-r^2}{1+r^2}, \qquad \sin t = \frac{2r}{1+r^2}, \qquad \frac{1-\cos t}{\sin t} = r,$$

  and the transitive dictionary $\frac{1}{2}(1 + \cos t + \sin t) = (1 + r)/(1 + r^2)$ derived from it. The gluing lemma and the subadditivity corollary, the latter in the form "ψ concave and nondecreasing with $\psi(0) = 0$ and $x \le y + z$ imply $\psi(x) \le \psi(y) + \psi(z)$". The extension $\widehat{E}$ of the threshold excess built from those two. The solution of the certificate system carried out there. The cubic bound $3 - 2r + 2r^2 - r^3 \ge 1$ on $[0, 1]$ and its corollary $K > 0$.
- From Chapter 5: the projected half-width $w_F(n)$ and the threshold of a branch with unit normal n, namely $H(F, G; n) = w_F(n) + w_G(n)$. The owned-branch threshold lemma: if n is, up to sign, an axis of one of the two frames then

$$H(F, G; n) = H\big(d_4(F, G)\big), \qquad H(x) = \frac{1 + \cos x + \sin x}{2} \;\ge\; 1,$$

  independently of which of the two frames supplies the axis. The frame circle $\mathbb{R}/(\pi/2)\mathbb{Z}$, on which a frame is the common direction of its axes modulo quarter turns, with metric d_4 taking values in $[0, \pi/4]$ and satisfying the triangle inequality. The corollary that two axes of one frame are either parallel or perpendicular.
- From Chapter 9: the freeze, and complete contact $\langle d_i, n_i\rangle = H_i$ for $i = 1, 2, 3$ at the minimiser.
- From Chapter 10: the oriented sines $s_1 = \sigma \det(n_1, n_2)$, $s_2 = \sigma \det(n_2, n_3)$, $s_3 = \sigma \det(n_3, n_1)$, and the *sine system*

$$H_1 = \lambda_2 s_1 + \lambda_3 s_3, \qquad H_2 = \lambda_3 s_2 + \lambda_1 s_1, \qquad H_3 = \lambda_1 s_3 + \lambda_2 s_2. \tag{12.1}$$

  The relabelling conventions, together with the check that they preserve the displayed form of (12.1). The dichotomy of the owner tournament into cyclic and transitive. The reduction of the centre-area lemma to the inequality $\lambda_1 H_1 + \lambda_2 H_2 + \lambda_3 H_3 \ge 2$.

Standing hypotheses

The data $(F_i, n_i, H_i, \sigma, \text{owners})$ fixed by the freeze of Chapter 9 are constants for the whole chapter. The configuration $x = (P_2, P_3) \in \mathbb{R}^4$ is a minimiser of the relaxed problem, with doubled area

$0 < D \le D_0 < 1$ and acute centre triangle; all three frozen branches are active there, $\langle d_i, n_i\rangle = H_i$; and Chapter 10 has produced multipliers $\lambda_1, \lambda_2, \lambda_3 \ge 0$ satisfying the sine system (12.1) together with $2D = \lambda_1 H_1 + \lambda_2 H_2 + \lambda_3 H_3$. We assume throughout that

$$s_1 > 0, \qquad s_2 > 0, \qquad s_3 > 0,$$

the nonpositive-sine case having been disposed of in Chapter 10, and we assume that the owner tournament is transitive. By the relabelling conventions of Chapter 10 the vertices may then be named so that

$$P_1 = \text{source}, \qquad P_2 = \text{middle}, \qquad P_3 = \text{sink}$$

in that order, with (12.1) and the contact equalities keeping their displayed form.

A remark on the freeze is needed. The relabelling above is a single element of S_3, and by the transformation table of Chapter 10 an odd element replaces σ by $-\sigma$ (and simultaneously reverses all three branches, which is why the displayed form survives). So σ as used below is the orientation sign of the relabelled configuration, not necessarily the sign frozen in Chapter 9. This affects nothing here: every argument below uses only $\sigma^2 = 1$, the identity $\langle J_\sigma u, v\rangle = \sigma \det(u, v)$ for whichever sign is in force, and the sign conventions of the sine system, which the transformation table shows to be preserved.

Throughout, "active branch" means equality in a chosen frozen linear inequality, not geometric tangency of squares; no square is moved or rotated.

Notation for this chapter

The following notation is local to this chapter.

- A, B, C denote the thresholds H_1, H_2, H_3; they are used for nothing else here.
- $E(x) = H(x) - 1$ is the threshold excess.
- The owned-branch threshold function is written $H(x)$, with argument x, rather than $H(t)$ as in Chapter 5: the letter t is needed throughout this chapter for the angle of Lemma 12.3. The argument of H is a frame distance.
- K is the certificate constant of Section 12.8.

The argument below runs through an algebraic identity rather than a chain of inequalities: the quantity to be bounded, minus 2, is written as a nonnegative combination of quantities already known to be nonnegative, plus a nonnegative constant. Once the identity is written down the inequality follows in one line.

12.1 The source owns two branches

Recall the convention of Chapter 10: each edge of the centre triangle is directed away from the owner of its branch, and the resulting tournament on three vertices is either a directed 3-cycle or a transitive order. In the transitive case, with the labelling fixed above, the source P_1 owns two branches:

- the branch of the pair $\{P_1, P_2\}$, with normal n_1, is owned by P_1, so n_1 is, up to sign, an axis of F_1;
- the branch of the pair $\{P_3, P_1\}$, with normal n_3, is also owned by P_1, so n_3 is, up to sign, an axis of F_1;
- the branch of the pair $\{P_2, P_3\}$, with normal n_2, is owned by the middle vertex P_2, so n_2 is, up to sign, an axis of F_2.

The sink P_3 owns nothing: the frame F_3 supplies none of the three normals. This asymmetry drives the rest of the section. Section 12.3 shows that H_3 nevertheless depends on F_3 and F_1 jointly.

Lemma 12.1 (Perpendicular source normals)**.** *In the transitive case $n_1 \perp n_3$ and $s_3 = 1$. Consequently (n_1, n_3) is an orthonormal basis of $\mathbb{R}^2$ with $\det(n_3, n_1) = \sigma$, and*

$$J_\sigma n_1 = -n_3.$$

Proof. Both n_1 and n_3 are, up to sign, axes of the single frame F_1. By Chapter 5, two axes of one frame are either parallel or perpendicular. If they were parallel then $n_3 = \pm n_1$, whence $\det(n_3, n_1) = 0$ and $s_3 = \sigma \det(n_3, n_1) = 0$, contradicting the standing hypothesis $s_3 > 0$. So $n_1 \perp n_3$. Two orthogonal unit vectors span a parallelogram of unit area, so $\det(n_3, n_1) = \pm 1$ and therefore $s_3 = \sigma \det(n_3, n_1) = \pm 1$; positivity forces $s_3 = 1$. Unwinding, $\sigma \det(n_3, n_1) = 1$ and hence $\det(n_3, n_1) = \sigma$, since $\sigma^2 = 1$.

For the last claim,

$$\langle J_\sigma n_1, n_3 \rangle = \sigma \det(n_1, n_3) = -\sigma \det(n_3, n_1) = -\sigma^2 = -1.$$

Both $J_\sigma n_1$ and n_3 are unit vectors, and two unit vectors with inner product -1 are negatives of one another, since $\|J_\sigma n_1 + n_3\|^2 = 1 + 2(-1) + 1 = 0$. Hence $J_\sigma n_1 = -n_3$. □

Remark 12.2. The reduction just performed is discrete: no metric fact about the squares was used, only the combinatorial statement that one square is the owner of two of the three frozen branches. Ownership carries no metric information, only the discrete information used here; this is why the transitive case is a one-parameter problem while the cyclic case of Chapter 11 is a three-parameter one.

12.2 One parameter: expanding n_2

Lemma 12.3 (The angle t)**.** *There is a unique $t \in (0, \pi/2)$ with*

$$s_3 = 1, \qquad s_1 = c := \cos t, \qquad s_2 = s := \sin t.$$

Moreover $c = \langle J_\sigma n_1, n_2 \rangle$, so t is the angle from $J_\sigma n_1$ to n_2.

Proof. By Lemma 12.1 we may write $n_2 = a\, n_1 + b\, n_3$ with $a = \langle n_2, n_1 \rangle$ and $b = \langle n_2, n_3 \rangle$. Since det is bilinear and alternating, $\det(n_1, n_1) = \det(n_3, n_3) = 0$ and therefore

$$\begin{aligned} \det(n_1, n_2) &= b \det(n_1, n_3) = -b \det(n_3, n_1) = -b\sigma, \\ \det(n_2, n_3) &= a \det(n_1, n_3) = -a\sigma. \end{aligned}$$

Multiplying by σ and using $\sigma^2 = 1$,

$$s_1 = \sigma \det(n_1, n_2) = -b, \qquad s_2 = \sigma \det(n_2, n_3) = -a.$$

As (n_1, n_3) is orthonormal, $s_1^2 + s_2^2 = a^2 + b^2 = \|n_2\|^2 = 1$. The standing hypothesis gives $s_1, s_2 > 0$, so (s_1, s_2) lies on the open arc of the unit circle in the first quadrant, which is parametrised bijectively by $t \in (0, \pi/2)$ via $(\cos t, \sin t)$. Uniqueness of t is the injectivity of that parametrisation. Finally $\langle J_\sigma n_1, n_2 \rangle = \sigma \det(n_1, n_2) = s_1 = c$, and since $J_\sigma n_1$ and n_2 are unit vectors this says exactly that the angle between them is t. □

Remark 12.4. The proof gives $a = -s < 0$ and $b = -c < 0$: in the basis (n_1, n_3) both coordinates of n_2 are strictly negative, so n_2 lies in the open quadrant spanned by $-n_1$ and $-n_3$. See Figure 12.1.

12.3 The thresholds A, B, C

Write

$$A = H\big(d_4(F_1, F_2)\big), \qquad B = H\big(d_4(F_2, F_3)\big), \qquad C = H\big(d_4(F_3, F_1)\big),$$

where $H(x) = (1 + \cos x + \sin x)/2$ is the owned-branch threshold function of Chapter 5. Note that the three arguments are the three pairwise frame distances, taken in the cyclic order F_1F_2, F_2F_3, F_3F_1; d_4 is symmetric, so the order within each pair is immaterial.

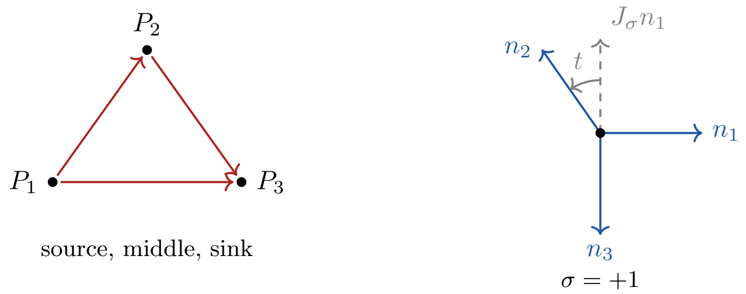


Figure 12.1: The transitive case. Left: the owner tournament, each red arrow pointing from the owner of a branch to the other endpoint of that edge of the centre triangle; owner arrows are not normals. Right: the source-owned normals $n_1 \perp n_3$, with n_2 at angle t from the grey guide $J_\sigma n_1 = -n_3$, so that n_2 lies in the quadrant spanned by $-n_1$ and $-n_3$.

Lemma 12.5 (Identification of the thresholds)**.** *$H_1 = A$, $H_2 = B$ and $H_3 = C$, and*

$$1 \ \le\ A, B, C \ \le\ \frac{1+\sqrt{2}}{2}.$$

Proof. Each of the three frozen branches is owned, so its normal is, up to sign, an axis of one of the two frames of its pair, and the owned-branch threshold lemma of Chapter 5 applies. That lemma computes the threshold as $H(d_4(F, F'))$, where F, F' are the two frames of the pair, and its content is that the answer does not record which of the two frames supplied the axis.

Branch 1 belongs to the pair $\{P_1, P_2\}$, so $H_1 = H(d_4(F_1, F_2)) = A$. Branch 2 belongs to the pair $\{P_2, P_3\}$, so $H_2 = H(d_4(F_2, F_3)) = B$. Branch 3 belongs to the pair $\{P_3, P_1\}$ and is owned by P_1, so its normal n_3 is an axis of F_1 and has nothing to do with F_3; the threshold of that branch is nevertheless $w_{F_3}(n_3) + w_{F_1}(n_3)$, a quantity built from both frames, and the owned-branch lemma evaluates it as $H(d_4(F_3, F_1)) = C$.

For the bounds, write $\cos x + \sin x = \sqrt{2}\cos(x - \pi/4)$. On $[0, \pi/4]$ the quantity $x - \pi/4$ runs over $[-\pi/4, 0]$, on which cos increases from $1/\sqrt{2}$ to 1; hence $\cos x + \sin x$ increases from 1 to $\sqrt{2}$ and H increases from 1 to $(1+\sqrt{2})/2$. Since d_4 takes values in $[0, \pi/4]$, the displayed bounds follow. $\square$

The threshold A is determined by the angle t of Lemma 12.3; the other two are not, and enter below only through inequalities.

Lemma 12.6 (A in terms of t)**.** *$d_4(F_1, F_2) = \min\{t, \frac{\pi}{2} - t\}$, and*

$$A = \frac{1 + c + s}{2}.$$

Proof. By Lemma 12.1, $J_\sigma n_1 = -n_3$. Both n_1 and n_3 are axes of F_1 up to sign and they are perpendicular, so the axis set of F_1 is exactly $\{\pm n_1, \pm n_3\} = \{\pm n_1, \pm J_\sigma n_1\}$. Consequently the point of the frame circle $\mathbb{R}/(\pi/2)\mathbb{Z}$ determined by F_1 is the direction of n_1 reduced modulo $\pi/2$, and it is equally the direction of $J_\sigma n_1$ reduced modulo $\pi/2$, the two differing by a quarter turn.

Since n_2 is an axis of F_2, the point of the frame circle determined by F_2 is the direction of n_2 modulo $\pi/2$. By Lemma 12.3 the direction of n_2 differs from that of $J_\sigma n_1$ by t measured in the σ-orientation, hence from that of n_1 by $\pi/2 + t$ in that orientation; either way the difference of the two points of the frame circle is the residue of t modulo $\pi/2$. (The frame circle carries the standard orientation, so if $\sigma = -1$ the residue is that of $-t$, that is $\pi/2 - t$; since $\min\{\pi/2 - t, t\}$ is symmetric in the two, the conclusion below is unchanged.) The distance on $\mathbb{R}/(\pi/2)\mathbb{Z}$ from the residue 0 to the residue of t, for $t \in (0, \pi/2)$, is $\min\{t, \pi/2 - t\}$. This proves the first claim, and shows in particular that $d_4(F_1, F_2)$ need not equal t.

For the second claim, $\cos x + \sin x = \sqrt{2}\cos(x - \pi/4)$ is invariant under $x \mapsto \pi/2 - x$, because that substitution replaces $x - \pi/4$ by its negative and cos is even. Hence $H(\pi/2 - x) = H(x)$, and

$$A = H\big(\min\{t, \tfrac{\pi}{2} - t\}\big) = H(t) = \frac{1 + \cos t + \sin t}{2} = \frac{1 + c + s}{2}. \qquad \square$$

The symbol $H(t)$ in the last display is the raw formula $(1+\cos t+\sin t)/2$ evaluated at t, which may exceed $\pi/4$. It is not the value at t of the extended excess $\widehat{E}$ of Section 12.5 plus 1: that extension is constant beyond $\pi/4$, whereas the raw formula continues to decrease. The two agree through the frame distance: always $A-1=\widehat{E}(d_4(F_1,F_2))$, while $A-1=\widehat{E}(t)$ only when $t\le\pi/4$.

Remark 12.7. $A>1$ for every $t\in(0,\pi/2)$, since $c+s=\sqrt{2}\cos(t-\pi/4)>1$ exactly when $|t-\pi/4|<\pi/4$, that is when $t\in(0,\pi/2)$. Also $A\le(1+\sqrt{2})/2$, with equality if and only if $t=\pi/4$. So in the transitive case the source–middle threshold is never at the universal minimum 1; the degenerate configurations sit on the boundary, at $t\to0$ and $t\to\pi/2$. Chapter 13 returns to that boundary when it discusses sharpness.

12.4 The transitive system and the target

Substituting $s_3=1$, $s_1=c$, $s_2=s$ from Lemma 12.3 and $H_1=A$, $H_2=B$, $H_3=C$ from Lemma 12.5 into the sine system (12.1) gives the *transitive system*

$$A=c\lambda_2+\lambda_3,\qquad B=c\lambda_1+s\lambda_3,\qquad C=\lambda_1+s\lambda_2. \tag{12.2}$$

Everything below is a consequence of (12.2), of the sign conditions $\lambda_i\ge0$ and $B,C\ge1$, and of one further inequality proved in Section 12.5. Which triples (t,λ_1,λ_2) are realised by actual configurations of squares is not decided here.

The goal is to prove that $A\lambda_1+B\lambda_2+C\lambda_3\ge2$.

Remark 12.8. Since $B\ge1$, $C\ge1$ and $\lambda_2,\lambda_3\ge0$,

$$A\lambda_1+B\lambda_2+C\lambda_3\ \ge\ A\lambda_1+\lambda_2+\lambda_3,$$

so it suffices to prove the formally stronger statement $A\lambda_1+\lambda_2+\lambda_3\ge2$.

That is not the only possible target. The cyclic case of Chapter 11 reaches the unweighted bound $\lambda_1+\lambda_2+\lambda_3\ge2$ and only then multiplies by $H_i\ge1$. The unweighted bound holds here as well, certified by the same two slacks; Exercise 12.6 asks for the derivation. We carry the weight A because it makes the coefficient-matching system of Section 12.7 the one already solved in Chapter 3, so that no new 2×2 solve is needed.

12.5 The frame-excess inequality

The first ingredient is a triangle inequality among the thresholds, measured by their excess over the universal bound $H\ge1$. Write

$$E(x)=H(x)-1=\frac{\cos x+\sin x-1}{2}\qquad(0\le x\le\pi/4)$$

for the threshold excess, and let $\widehat{E}\colon[0,\infty)\to\mathbb{R}$ be its extension by the constant value $E(\pi/4)=(\sqrt{2}-1)/2$ beyond $\pi/4$, as constructed in Chapter 3. The extension is needed because the proof below evaluates $\widehat{E}$ at a sum of two frame distances, and a sum of two numbers in $[0,\pi/4]$ can reach $\pi/2$.

Lemma 12.9 (The excess function)**.** *$\widehat{E}$ is C^1, nonnegative, nondecreasing and concave on $[0,\infty)$, with $\widehat{E}(0)=0$. Consequently, for $x,y,z\ge0$,*

$$x\le y+z\quad\Longrightarrow\quad\widehat{E}(x)\le\widehat{E}(y)+\widehat{E}(z).$$

Proof. Chapter 3 computes $E(0)=0$, $E'(x)=(\cos x-\sin x)/2$ and $E''(x)=-(\cos x+\sin x)/2<0$ on $[0,\pi/4]$, so that E is concave with E' nonincreasing and $E'(\pi/4)=0$. Its gluing lemma, applied with $a=\pi/4$, is the assertion that the constant extension $\widehat{E}$ inherits all four listed properties. The displayed implication is its subadditivity corollary, applied to $\psi=\widehat{E}$. ☐

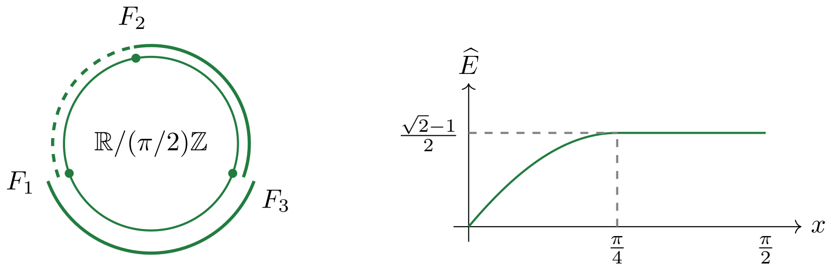


Figure 12.2: Left: the frame circle $\mathbb{R}/(\pi/2)\mathbb{Z}$ with the three frames of the transitive case. The distance d_4 between two of them is the length of the shorter of the two arcs joining them, so it is not always the naive difference of angles, although it is whenever that difference is at most $\pi/4$, as it is for all three arcs drawn here. The dashed arc realises $d_4(F_1, F_2)$ and the two solid arcs realise $d_4(F_2, F_3)$ and $d_4(F_3, F_1)$; the triangle inequality says the dashed arc is no longer than the two solid arcs together. Right: the excess $\widehat{E}$, drawn with the vertical axis stretched relative to the horizontal, strictly concave on $[0, \pi/4]$ and constant thereafter, the join being smooth because $E'(\pi/4) = 0$.

Lemma 12.10 (Frame excess)**.** *For any three frames F_1, F_2, F_3,*

$$H\big(d_4(F_1, F_2)\big) + 1 \;\le\; H\big(d_4(F_2, F_3)\big) + H\big(d_4(F_3, F_1)\big);$$

in particular, for the three frames of the transitive case,

$$A + 1 \;\le\; B + C.$$

Proof. By Chapter 5, d_4 is a metric on the frame circle $\mathbb{R}/(\pi/2)\mathbb{Z}$, so it obeys the triangle inequality

$$d_4(F_1, F_2) \;\le\; d_4(F_1, F_3) + d_4(F_3, F_2).$$

All three frame distances lie in $[0, \pi/4]$, where $\widehat{E}$ agrees with E; so, using the symmetry of d_4,

$$A - 1 = \widehat{E}\big(d_4(F_1, F_2)\big), \quad B - 1 = \widehat{E}\big(d_4(F_3, F_2)\big), \quad C - 1 = \widehat{E}\big(d_4(F_1, F_3)\big).$$

Lemma 12.9, applied with $x = d_4(F_1, F_2)$, $y = d_4(F_1, F_3)$ and $z = d_4(F_3, F_2)$, gives $A - 1 \le (C - 1) + (B - 1)$, which rearranges to the claim. $\square$

Remark 12.11. The inequality $A \le B + C$ holds trivially, since $A \le (1 + \sqrt{2})/2 < 2 \le B + C$ by Lemma 12.5. The content of Lemma 12.10 lies in the additive constant 1, that is, in measuring thresholds by their excess over the universal bound rather than by their raw size. The excess is what obeys a triangle inequality; the threshold itself does not, in any useful sense.

Example 12.12 (Frame distances and the wrap-around)**.** Let F_1, F_2, F_3 be frames whose axes point in the directions 0, $\pi/6$ and $\pi/3$ respectively, all read modulo $\pi/2$. The three frame distances are computed by taking the difference of the directions modulo $\pi/2$ and then the smaller of the residue and its complement:

$$d_4(F_1, F_2) = \min\{\tfrac{\pi}{6}, \tfrac{\pi}{3}\} = \tfrac{\pi}{6}, \qquad d_4(F_2, F_3) = \min\{\tfrac{\pi}{6}, \tfrac{\pi}{3}\} = \tfrac{\pi}{6},$$

and

$$d_4(F_3, F_1) = \min\{\tfrac{\pi}{3}, \tfrac{\pi}{2} - \tfrac{\pi}{3}\} = \tfrac{\pi}{6}, \qquad \text{not } \tfrac{\pi}{3}.$$

Hence all three thresholds equal

$$H(\tfrac{\pi}{6}) = \frac{1 + \frac{\sqrt{3}}{2} + \frac{1}{2}}{2} = \frac{3 + \sqrt{3}}{4} \approx 1.1830,$$

and the frame-excess inequality reads $A + 1 \approx 2.1830 \le B + C \approx 2.3660$; by symmetry it holds for each of the three choices of which threshold plays the role of A.

Now a second instance, in which the extension of E past $\pi/4$ is used and which is compatible with the standing hypotheses of this chapter. Take frames with directions 0, $\pi/8$, $\pi/4$ in the roles F_1, F_2, F_3. Then

$$d_4(F_1, F_2) = \tfrac{\pi}{8}, \qquad d_4(F_3, F_2) = \tfrac{\pi}{8}, \qquad d_4(F_1, F_3) = \tfrac{\pi}{4}.$$

The middle term of the chain in the proof of Lemma 12.10 is therefore

$$\widehat{E}\big(d_4(F_1, F_3) + d_4(F_3, F_2)\big) = \widehat{E}(3\pi/8),$$

a quantity defined only by the extension. Here

$$A = B = H(\tfrac{\pi}{8}) = \frac{1 + \cos\frac{\pi}{8} + \sin\frac{\pi}{8}}{2} \approx 1.1533,$$

$$C = H(\tfrac{\pi}{4}) = \frac{1 + \sqrt{2}}{2} \approx 1.2071,$$

and the inequality reads $A + 1 \approx 2.1533 \leq B + C \approx 2.3604$. Note that $A > 1$, as Remark 12.7 requires of any transitive configuration; the value $d_4(F_1, F_2) = \pi/8$ corresponds to $t = \pi/8$ or $t = 3\pi/8$, both admissible.

12.6 Dual certificates

Suppose one must prove $T \geq 2$, where T is a linear expression in some unknowns, the unknowns satisfy some linear equations, and certain expressions $S_1, \ldots, S_k$ in the unknowns, called the *slacks*, are known to be nonnegative. A *dual certificate* is an identity

$$T - 2 = \alpha_1 S_1 + \cdots + \alpha_k S_k + K, \tag{12.3}$$

valid modulo the equations, in which every $\alpha_j \geq 0$ and $K \geq 0$ is a constant, that is, free of the unknowns. Given such an identity the inequality $T \geq 2$ is immediate. Verifying (12.3) requires only the expansion of polynomials, with no knowledge of where the coefficients came from.

How the coefficients are found. The α_j are forced by the requirement that the right-hand side reproduce the unknown-dependence of the left. Use the equations to eliminate as many unknowns as possible; then, in the surviving unknowns, match the coefficient of each. That is a small linear system in $(\alpha_1, \ldots, \alpha_k)$: as many equations as surviving unknowns. Whatever is left over after the matching is the constant K; only its sign can fail. If $K < 0$ then the chosen slack list is too weak, and one must find another true inequality to add to it, or else the claim is false.

The dual-programme reading. Consider the problem of minimising T subject to the linear equations and the inequalities $S_j \geq 0$. The vector $(\alpha_1, \ldots, \alpha_k, K)$ is a feasible solution of the dual of that linear programme, and the implication "identity $\Rightarrow$ bound" is what optimisation calls weak duality. We do not develop this. The one duality theorem proved in this book is the Farkas lemma of Chapter 8: there the objective gradient is written as a nonnegative combination of the active constraint normals, and here the objective is written as a nonnegative combination of the active inequalities themselves.

Example 12.13 (A certificate in two variables)**.** Let $x, y \geq 0$ satisfy $x + 2y = 3$, and suppose the slack $x - 1 \geq 0$ is known. Eliminating y by $y = (3 - x)/2$ gives $x + y = (x + 3)/2$, and so

$$x + y - 2 = \frac{x + 3}{2} - 2 = \frac{x - 1}{2}.$$

This is a certificate with $\alpha = \frac{1}{2}$ and $K = 0$; hence $x + y \geq 2$.

Vary the target. To prove $x + y \geq \frac{3}{2}$ the same elimination gives

$$x + y - \tfrac{3}{2} = \frac{x}{2} = \frac{x - 1}{2} + \frac{1}{2},$$

so $\alpha = \frac{1}{2}$ and $K = \frac{1}{2} \geq 0$: again a valid certificate. To prove $x + y \geq \frac{5}{2}$ it gives

$$x + y - \tfrac{5}{2} = \frac{x-2}{2} = \frac{x-1}{2} - \frac{1}{2},$$

so $\alpha = \frac{1}{2}$ and $K = -\frac{1}{2} < 0$, and no certificate is obtained. None exists: at $x = 1$, $y = 1$ all hypotheses hold and $x + y = 2 < \frac{5}{2}$. A third slack, $y \geq 0$, was available throughout and would not have helped. A failure of the recipe says only that the chosen slacks do not suffice.

12.7 Deriving the coefficients

We now build the certificate for the transitive system (12.2). The target is $T = A\lambda_1 + \lambda_2 + \lambda_3$ and the goal is $T \geq 2$. Four nonnegative quantities are available: $A - 1$, $B - 1$, $C - 1$ from Lemma 12.5, and $B + C - A - 1$ from Lemma 12.10. Two unknowns will survive the elimination below, so by the counting rule of Section 12.6 we must choose two. We take

$$S_1 = C - 1 \ \geq \ 0 \quad \text{(Lemma 12.5)}, \qquad S_2 = B + C - A - 1 \ \geq \ 0 \quad \text{(Lemma 12.10)}.$$

Exercise 12.8 runs the same recipe on the pair $(B - 1, C - 1)$ instead and finds that it fails for t close to $\pi/2$.

Elimination

Use the *first* equation of (12.2) only, in the form $\lambda_3 = A - c\lambda_2$; the second and third equations enter solely through B and C inside the slacks. Then

$$T - 2 = A\lambda_1 + \lambda_2 + (A - c\lambda_2) - 2 = A\lambda_1 + (1 - c)\lambda_2 + (A - 2), \tag{12.4}$$

$$S_1 = C - 1 = \lambda_1 + s\lambda_2 - 1, \tag{12.5}$$

and, since $B = c\lambda_1 + s\lambda_3 = c\lambda_1 - sc\lambda_2 + sA$,

$$S_2 = B + C - A - 1 = (1 + c)\lambda_1 + s(1 - c)\lambda_2 + (s - 1)A - 1. \tag{12.6}$$

Coefficient matching

Demand that

$$(T - 2) - \alpha S_1 - \beta S_2$$

be free of λ_1 and of λ_2. Reading off the two coefficients from (12.4)–(12.6) gives the linear system

$$\alpha + (1 + c)\beta = A, \qquad s\alpha + s(1 - c)\beta = 1 - c. \tag{12.7}$$

Its matrix $\begin{pmatrix} 1 & 1+c \\ s & s(1-c) \end{pmatrix}$ has determinant $s(1 - c) - s(1 + c) = -2sc$, which is nonzero because t lies in the open interval $(0, \pi/2)$, so that $s > 0$ and $c > 0$. Hence (α, β) exists and is unique. Openness of the interval is the standing hypothesis $s_1, s_2 > 0$ of Chapter 10, transported through Lemma 12.3.

Divide the second equation of (12.7) by $s > 0$ and use the half-angle identity of Chapter 3,

$$\frac{1 - c}{s} = \frac{1 - \cos t}{\sin t} = \tan\frac{t}{2} = r,$$

to obtain

$$\alpha + (1 + c)\beta = A, \qquad \alpha + (1 - c)\beta = r. \tag{12.8}$$

Solving

Throughout, $r = \tan(t/2)$, which lies in $(0,1)$ because $t \in (0,\pi/2)$ and tan is increasing with $\tan(\pi/4) = 1$. Record the standard consequences

$$c = \frac{1-r^2}{1+r^2}, \quad s = \frac{2r}{1+r^2}, \quad 1+c = \frac{2}{1+r^2}, \quad 1-c = \frac{2r^2}{1+r^2}, \quad 1-s = \frac{(1-r)^2}{1+r^2}, \tag{12.9}$$

and the transitive dictionary of Chapter 3, used repeatedly below,

$$A = \frac{1+c+s}{2} = \frac{1+r}{1+r^2}. \tag{12.10}$$

The system (12.7), with A written out as $(1+c+s)/2$, is the 2×2 system solved in Chapter 3, whose elimination we do not repeat. Its unique solution is

$$\alpha = \frac{r}{(1+r)(1+r^2)}, \qquad \beta = \frac{1+r+r^2}{2(1+r)}. \tag{12.11}$$

Substituting (12.11) into (12.8) confirms it directly: $\alpha+(1+c)\beta$ has numerator $r+(1+r+r^2) = (1+r)^2$ over $(1+r)(1+r^2)$, which is A; and $\alpha+(1-c)\beta$ has numerator $r+r^2(1+r+r^2) = r(1+r)(1+r^2)$ over the same denominator, which is r.

The leftover constant

With α, β as above, the expression $(T-2)-\alpha S_1 - \beta S_2$ has no λ_1 and no λ_2 in it. What survives is read off from the constant terms of (12.4)–(12.6): the target contributes $A-2$, the first slack contributes $-\alpha\cdot(-1) = +\alpha$, and the second contributes $-\beta\big((s-1)A-1\big) = \beta + (1-s)A\beta$. Hence

$$K = (A-2) + \alpha + \beta + (1-s)A\beta. \tag{12.12}$$

We evaluate (12.12) in four steps. First,

$$\begin{aligned}\alpha+\beta &= \frac{r}{(1+r)(1+r^2)} + \frac{1+r+r^2}{2(1+r)} = \frac{2r+(1+r+r^2)(1+r^2)}{2(1+r)(1+r^2)} \\ &= \frac{1+3r+2r^2+r^3+r^4}{2(1+r)(1+r^2)},\end{aligned}$$

and since $(1+r)(r^3+2r+1) = r^4+r^3+2r^2+3r+1$ the numerator has the factor $1+r$, so

$$\alpha+\beta = \frac{r^3+2r+1}{2(1+r^2)}. \tag{12.13}$$

Second, using $1-s = (1-r)^2/(1+r^2)$ from (12.9), $A = (1+r)/(1+r^2)$ and $(1+r+r^2)(1-r) = 1-r^3$,

$$\begin{aligned}(1-s)A\beta &= \frac{(1-r)^2}{1+r^2}\cdot\frac{1+r}{1+r^2}\cdot\frac{1+r+r^2}{2(1+r)} \\ &= \frac{(1-r)^2(1+r+r^2)}{2(1+r^2)^2} = \frac{(1-r)(1-r^3)}{2(1+r^2)^2}.\end{aligned} \tag{12.14}$$

Third,

$$A-2 = \frac{1+r}{1+r^2} - 2 = \frac{1+r-2-2r^2}{1+r^2} = -\frac{2r^2-r+1}{1+r^2}. \tag{12.15}$$

Fourth, put (12.13)–(12.15) over the common denominator $2(1+r^2)^2$. The numerator is

$$\begin{aligned}N &= (1+r^2)\big[(r^3+2r+1) - 2(2r^2-r+1)\big] + (1-r)(1-r^3) \\ &= (1+r^2)(r^3-4r^2+4r-1) + (r-1)^2(r^2+r+1),\end{aligned}$$

where the second term was rewritten using $(1-r)(1-r^3) = (r-1)(r^3-1) = (r-1)^2(r^2+r+1)$. Since $r^3 - 4r^2 + 4r - 1 = (r-1)(r^2-3r+1)$ and $(r-1)^2(r^2+r+1) = (r-1)(r^3-1)$, the factor $r-1$ comes out:

$$N = (r-1)\Big[(1+r^2)(r^2-3r+1) + (r^3-1)\Big].$$

The bracket expands to $(r^4-3r^3+2r^2-3r+1)+(r^3-1) = r^4-2r^3+2r^2-3r = r(r^3-2r^2+2r-3)$, so

$$N = (r-1)\,r\,(r^3-2r^2+2r-3) = r(1-r)(3-2r+2r^2-r^3),$$

the last equality because both factors changed sign. We have proved that for $t \in (0,\pi/2)$ and $r = \tan(t/2) \in (0,1)$,

$$\alpha = \frac{r}{(1+r)(1+r^2)}, \qquad \beta = \frac{1+r+r^2}{2(1+r)}, \qquad K = \frac{r(1-r)\big(3-2r+2r^2-r^3\big)}{2(1+r^2)^2}. \tag{12.16}$$

12.8 The certificate

Lemma 12.14 (Transitive dual certificate)**.** *Let $t \in (0,\pi/2)$, and set $c = \cos t$, $s = \sin t$, $r = \tan(t/2) \in (0,1)$ and $A = (1+c+s)/2$. Let $\lambda_1, \lambda_2, \lambda_3, B, C$ be any real numbers satisfying the transitive system (12.2). Then, with α, β, K as in (12.16),*

$$A\lambda_1 + \lambda_2 + \lambda_3 - 2 = \alpha\,(C-1) + \beta\,(B+C-A-1) + K. \tag{12.17}$$

Moreover $\alpha > 0$, $\beta > 0$ and $K > 0$.

Proof. Substituting $\lambda_3 = A - c\lambda_2$ from the first equation of (12.2), and B, C from its second and third equations, the computations (12.4)–(12.6) of Section 12.7 give, for any real λ_1, λ_2,

$$\begin{aligned}
&\big(A\lambda_1 + \lambda_2 + \lambda_3 - 2\big) - \alpha(C-1) - \beta(B+C-A-1) \\
&\quad = \big[A - \alpha - (1+c)\beta\big]\lambda_1 + \big[(1-c) - s\alpha - s(1-c)\beta\big]\lambda_2 \\
&\qquad + \big[(A-2) + \alpha + \beta + (1-s)A\beta\big].
\end{aligned}$$

The coefficient of λ_1 vanishes by the first equation of (12.8), which Section 12.7 verifies directly for the data (12.16). The coefficient of λ_2 vanishes as well: by the second equation of (12.8) we have $\alpha + (1-c)\beta = r$, and multiplying by s and using $sr = 1-c$ gives $s\alpha + s(1-c)\beta = sr = 1-c$. The remaining bracket is exactly the constant (12.12), which the computation of Section 12.7 evaluated to the displayed K. This proves (12.17).

Positivity of α and β is by inspection: for $r \in (0,1)$ every factor in (12.16) is positive. For K, the prefactor $r(1-r)$ and the denominator $2(1+r^2)^2$ are positive on $(0,1)$, so only the cubic matters, and Chapter 3 shows $3-2r+2r^2-r^3 \ge 1$ on $[0,1]$ by the regrouping $3-2r+2r^2-r^3 = 2(1-r)+1+r^2(2-r)$. Hence $K > 0$ on $(0,1)$. □

Note two features of (12.17). It is an identity: it holds for all real $\lambda_1, \lambda_2, \lambda_3, B, C$ subject to (12.2), with no inequality hypotheses. And the multipliers appear on the right only inside the two slacks, so the geometry and analysis of Chapter 5 and Lemma 12.10 supply the signs of the slacks while the algebra of Section 12.7 supplies the identity.

Corollary 12.15. *In the transitive case, $A\lambda_1 + \lambda_2 + \lambda_3 \ge 2$.*

Proof. Apply Lemma 12.14 to the actual data of the minimiser, which satisfies (12.2). On the right-hand side of (12.17) we have $\alpha, \beta > 0$ and $K > 0$, while $C - 1 \ge 0$ by Lemma 12.5 and $B + C - A - 1 \ge 0$ by Lemma 12.10. Hence the right-hand side is nonnegative, and so is the left. □

The checking recipe

The identity (12.17) may be verified independently by the following procedure.

(1) Form the difference of the two sides of (12.17).

(2) Substitute $B = c\lambda_1 + s\lambda_3$ and $C = \lambda_1 + s\lambda_2$ into the two slacks.

(3) Eliminate λ_3 by $\lambda_3 = A - c\lambda_2$, using the *first* equation of (12.2) and no other.

(4) Substitute $c = (1-r^2)/(1+r^2)$, $s = 2r/(1+r^2)$, $A = (1+r)/(1+r^2)$ and the formulas (12.16) for α, β, K.

(5) Every denominator divides $2(1+r)(1+r^2)^2$, which is positive; clear it. What remains is a polynomial in r, λ_1, λ_2, and it expands to 0.

Example 12.16 (The certificate at $t = \pi/4$)**.** Take $t = \pi/4$, so that $c = s = 1/\sqrt{2}$ and

$$r = \tan\frac{\pi}{8} = \sqrt{2} - 1, \qquad A = \frac{1+c+s}{2} = \frac{1+\sqrt{2}}{2}.$$

(As a check on (12.10): $1 + r^2 = 1 + (3 - 2\sqrt{2}) = 4 - 2\sqrt{2}$ and $1 + r = \sqrt{2}$, so $(1+r)/(1+r^2) = \sqrt{2}/(4-2\sqrt{2}) = \sqrt{2}(4+2\sqrt{2})/8 = (1+\sqrt{2})/2$, as it should be.)

Choose $\lambda_1 = \lambda_2 = 1/\sqrt{2}$. The transitive system (12.2) then determines the remaining data:

$$\lambda_3 = A - c\lambda_2 = \frac{1+\sqrt{2}}{2} - \frac{1}{2} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}},$$

$$B = c\lambda_1 + s\lambda_3 = \frac{1}{2} + \frac{1}{2} = 1, \qquad C = \lambda_1 + s\lambda_2 = \frac{1}{\sqrt{2}} + \frac{1}{2} = \frac{1+\sqrt{2}}{2} = A.$$

The certificate data: $(1+r)(1+r^2) = \sqrt{2}\,(4-2\sqrt{2}) = 4\sqrt{2} - 4 = 4(\sqrt{2}-1) = 4r$, so

$$\alpha = \frac{r}{4r} = \frac{1}{4} \quad \text{exactly,}$$
$$\beta = \frac{1+r+r^2}{2(1+r)} = \frac{1+(\sqrt{2}-1)+(3-2\sqrt{2})}{2\sqrt{2}} = \frac{3-\sqrt{2}}{2\sqrt{2}} = \frac{3\sqrt{2}-2}{4} \approx 0.5607.$$

For K, note $3 - 2r + 2r^2 - r^3$ with $r = \sqrt{2} - 1$: we have $r^2 = 3 - 2\sqrt{2}$ and $r^3 = r \cdot r^2 = (\sqrt{2}-1)(3-2\sqrt{2}) = 5\sqrt{2} - 7$, so

$$3 - 2r + 2r^2 - r^3 = 3 - (2\sqrt{2}-2) + (6-4\sqrt{2}) - (5\sqrt{2}-7) = 18 - 11\sqrt{2},$$

while $r(1-r) = (\sqrt{2}-1)(2-\sqrt{2}) = 3\sqrt{2}-4$ and $2(1+r^2)^2 = 2(4-2\sqrt{2})^2 = 2(24-16\sqrt{2}) = 48-32\sqrt{2}$. Hence

$$K = \frac{(3\sqrt{2}-4)(18-11\sqrt{2})}{48-32\sqrt{2}} = \frac{54\sqrt{2}-66-72+44\sqrt{2}}{48-32\sqrt{2}} = \frac{98\sqrt{2}-138}{48-32\sqrt{2}}$$
$$= \frac{49\sqrt{2}-69}{24-16\sqrt{2}} = \frac{9\sqrt{2}-11}{8} \approx 0.21599,$$

the last simplification on multiplying numerator and denominator by $24 + 16\sqrt{2}$ and using $24^2 - 16^2 \cdot 2 = 576 - 512 = 64$.

Now compare the two sides of (12.17). The slacks are

$$C - 1 = \frac{\sqrt{2}-1}{2} \approx 0.20711, \qquad B + C - A - 1 = 1 + A - A - 1 = 0 \quad \text{exactly.}$$

The left-hand side is

$$A\lambda_1 + \lambda_2 + \lambda_3 - 2 = \frac{1+\sqrt{2}}{2} \cdot \frac{1}{\sqrt{2}} + \frac{2}{\sqrt{2}} - 2$$
$$= \frac{\sqrt{2}+2}{4} + \sqrt{2} - 2 = \frac{5\sqrt{2}}{4} - \frac{3}{2} \approx 0.26777,$$

and the right-hand side is

$$\frac{1}{4}\cdot\frac{\sqrt{2}-1}{2}+\beta\cdot 0+\frac{9\sqrt{2}-11}{8}=\frac{\sqrt{2}-1}{8}+\frac{9\sqrt{2}-11}{8}=\frac{10\sqrt{2}-12}{8}=\frac{5\sqrt{2}}{4}-\frac{3}{2}.$$

The two sides agree.

Four comments. (i) The frame-excess slack vanishes here, $B+C-A-1=0$, so the right-hand side reduces to $\alpha(C-1)+K$. Note also $B=1$, so $B-1=0$ as well, and the two slacks are indistinguishable here. Exercise 12.8 shows that the alternative pair $(B-1, C-1)$ proves the bound for this data, and that Lemma 12.10 becomes indispensable only for t close to $\pi/2$, namely when $r^4+r^2>1$, that is $t \gtrsim 1.33$. (ii) The frame distances implied by $A=C=(1+\sqrt{2})/2$ and $B=1$ are $\pi/4$, 0, $\pi/4$, which satisfy the triangle inequality on the frame circle with equality, so the data is realised by actual frames. (iii) $A=C=(1+\sqrt{2})/2$ is the largest value a threshold can take, by Lemma 12.5. (iv) For this data $A\lambda_1+B\lambda_2+C\lambda_3=1+\sqrt{2}\approx 2.4142\geq 2$, so the corresponding minimiser would have $2D=1+\sqrt{2}$ and $D\approx 1.207\geq 1$.

Example 12.17 (Tightness as $t\to\pi/2$)**.** Fix $\lambda_1=\lambda_2=\frac{1}{2}$ and let $t\to\pi/2$, that is $r\to 1$. Then $c\to 0$, $s\to 1$, and by (12.10), $A\to 2/2=1$. The transitive system (12.2) gives

$$\lambda_3=A-c\lambda_2\to 1,\qquad B=c\lambda_1+s\lambda_3\to 1,\qquad C=\lambda_1+s\lambda_2\to\tfrac{1}{2}+\tfrac{1}{2}=1.$$

So both slacks tend to 0: $C-1\to 0$ and $B+C-A-1\to 1+1-1-1=0$. The constant tends to 0 as well, because of the factor $1-r$ in (12.16). And the target tends to

$$A\lambda_1+\lambda_2+\lambda_3\to\tfrac{1}{2}+\tfrac{1}{2}+1=2.$$

Hence the constant 2 is best possible for the system (12.2): no constant larger than 2 is a valid lower bound for $A\lambda_1+\lambda_2+\lambda_3$ over the solutions of that system with all the sign conditions in force. The certificate is tight in that limit: every nonnegative term on the right of (12.17) vanishes simultaneously. The statement concerns the system only. Which triples (t,λ_1,λ_2) are realised by actual configurations of squares is not determined here, so nothing above shows that 2 cannot be improved for data coming from a minimiser; that is a separate, geometric statement, settled in Chapter 13.

The limiting data is not itself an instance of the transitive case. At $t=\pi/2$ we would have $s_1=\cos t=0$, which the standing hypothesis $s_1>0$ excludes; that configuration is covered by the nonpositive-sine proposition of Chapter 10, not by anything in this chapter. Chapter 13 identifies the limit with three axis-parallel unit squares at three corners of a unit square, the configuration for which the centre-area lemma is sharp.

12.9 Closing the transitive case

Proposition 12.18 (Transitive case)**.** *In the transitive case, $\lambda_1H_1+\lambda_2H_2+\lambda_3H_3\geq 2$.*

Proof. By Lemma 12.5, $\lambda_1H_1+\lambda_2H_2+\lambda_3H_3=A\lambda_1+B\lambda_2+C\lambda_3$. Since $B\geq 1$, $C\geq 1$ and $\lambda_2,\lambda_3\geq 0$,

$$A\lambda_1+B\lambda_2+C\lambda_3\ \geq\ A\lambda_1+\lambda_2+\lambda_3\ \geq\ 2$$

by Corollary 12.15. □

Since Chapter 10 established $2D=\lambda_1H_1+\lambda_2H_2+\lambda_3H_3$ at the minimiser, Proposition 12.18 yields $D\geq 1$ in this case. That is $D\geq 1$ for the minimiser, which is not the statement $D_0\geq 1$ for the assumed counterexample. Chapter 13 makes that step, combines Proposition 12.18 with its cyclic and nonpositive-sine counterparts, and checks that the three cases are exhaustive.

Exercises

Exercise 12.1. Let n and n' be axes of a single frame F. Show that $\det(n,n')\in\{-1,0,1\}$, and that $\det(n,n')=0$ if and only if $n=\pm n'$. Deduce that in the transitive case $s_3\in\{-1,0,1\}$ before any use of positivity, and hence that the standing hypothesis $s_3 > 0$ forces $s_3 = 1$. Then exhibit a frame F_1, unit vectors n_1, n_3 that are axes of it, and a sign σ, for which $s_3 = -1$; and say which of the standing hypotheses of this chapter that example violates, and which of them it satisfies.

Exercise 12.2. Three frames have axis directions 0, $\pi/12$ and $5\pi/12$ modulo $\pi/2$. Compute the three frame distances and the three thresholds, and verify the frame-excess inequality for each of the three choices of which threshold plays the role of A. One of the three underlying triangle inequalities for d_4 is an equality; identify it, and explain why the corresponding frame-excess inequality is nevertheless strict. Then decide whether the proof of Lemma 12.10 evaluates $\widehat{E}$ outside $[0, \pi/4]$ for these frames, and exhibit a triple of frames for which it does.

Exercise 12.3. Show that in the transitive case $1 < A \leq (1 + \sqrt{2})/2$, with equality on the right exactly when $t = \pi/4$. Deduce that $d_4(F_1, F_2) \in (0, \pi/4]$, so that F_1 and F_2 are distinct points of the frame circle; in particular no instance of Lemma 12.10 with $d_4(F_1, F_2) = 0$ can arise from a transitive configuration, even though the lemma itself allows it. Show finally that A determines t exactly up to the involution $t \mapsto \pi/2 - t$, and that for $t \neq \pi/4$ the two values of t give the same A but different α, β and K (compute $\tan(\pi/4 - t/2)$ in terms of r to see how r transforms). Explain why this is not a contradiction, by identifying which part of the transitive system (12.2) fails to be symmetric in c and s.

Exercise 12.4. Go through Sections 12.5–12.9 and record, for each of the hypotheses $\lambda_1 \geq 0$, $\lambda_2 \geq 0$, $\lambda_3 \geq 0$, $B \geq 1$, $C \geq 1$ and $s_1, s_2 > 0$, every place at which it is used. Show in particular that $\lambda_1 \geq 0$ is used nowhere, and that $\lambda_2, \lambda_3 \geq 0$ and $B \geq 1$ are used at one step only, namely the first inequality in the proof of Proposition 12.18. Deduce that Corollary 12.15 remains true if the multipliers are allowed to be negative.

Exercise 12.5. Take $t = \pi/4$, so that α, β and K are as computed in Example 12.16, but take $\lambda_1 = 1$ and $\lambda_2 = \frac{1}{2}$. Compute λ_3, B and C from (12.2) in exact surds, check that $B, C \geq 1$ and $B + C \geq A + 1$, and verify identity (12.17) for this data. Then take $t = \pi/4$ again with $\lambda_1 = \lambda_2 = 0$, recompute λ_3, B and C from (12.2), and check that $B = (2 + \sqrt{2})/4 < 1$ and $C = 0$, so that both slacks are strictly negative. Verify that the identity nevertheless holds exactly, with both sides equal to $A - 2$. Explain why this is not a counterexample to Corollary 12.15, and say which of the two halves of the argument, the identity or the sign of the slacks, this data tests.

Exercise 12.6. Run the recipe of Section 12.7 again, with the same two slacks $C - 1$ and $B + C - A - 1$, but with the unweighted target $T_0 = \lambda_1 + \lambda_2 + \lambda_3$. Show that the matching system becomes

$$\alpha_0 + (1 + c)\beta_0 = 1, \qquad \alpha_0 + (1 - c)\beta_0 = r,$$

so that

$$\alpha_0 = \frac{r}{1 + r}, \qquad \beta_0 = \frac{1 + r^2}{2(1 + r)}, \qquad K_0 = \frac{r(1 - r)^2}{2(1 + r^2)},$$

and that all three are positive on $(0, 1)$. Deduce $\lambda_1 + \lambda_2 + \lambda_3 \geq 2$ in the transitive case, and hence Proposition 12.18 by the route the cyclic case of Chapter 11 takes, namely multiplying by $H_i \geq 1$ at the end. Finally show that $K_0 < K$ for every $r \in (0, 1)$, by checking that the difference of the two leftover constants has the factor $2 - r + r^2$, which has negative discriminant.

Exercise 12.7. Let $L = \pi/2$ be the circumference of the frame circle. Given three frames, let $g_1, g_2, g_3 \geq 0$ be the three gaps between consecutive frames as one travels once round the circle, so that $g_1 + g_2 + g_3 = L$, and note that the frame distance realised by a gap g is $\min\{g, L - g\}$.

(i) Show that at most one g_i can exceed $L/2 = \pi/4$.

(ii) Deduce that a triple $(a, b, c) \in [0, \pi/4]^3$ arises as the three pairwise frame distances of some three frames if and only if either $a + b + c = \pi/2$, or one of a, b, c equals the sum of the other two. Prove both directions.

(iii) Deduce Lemma 12.10 again, this time without any use of concavity or of $\widehat{E}$.

(iv) Decide whether three frames can have all three pairwise distances equal to $\pi/8$, and hence whether $A = B = C = H(\pi/8)$ can occur. Do the same for $A = B = C = H(\pi/6)$.

Exercise 12.8. Repeat the derivation of Section 12.7 with the slack pair $B-1$ and $C-1$ in place of $C-1$ and $B+C-A-1$. Show that the coefficients are forced to be

$$\frac{1+r+r^2}{2(1+r)} \quad (\text{of } B-1), \qquad \frac{r^3+2r+1}{2(1+r^2)} \quad (\text{of } C-1),$$

and that the leftover constant of this variant is

$$L = \frac{r(r-1)(r^4+r^2-1)}{(1+r)(1+r^2)^2}.$$

Show that $L \geq 0$ exactly when $r^4+r^2 \leq 1$, that is when $r^2 \leq (\sqrt{5}-1)/2$, and compute the corresponding threshold value of t to three decimal places. Conclude that the pair $(B-1,\, C-1)$ alone does not suffice for t near $\pi/2$. (This does not show that no argument avoiding Lemma 12.10 altogether exists.)

Chapter 13

Proof of the Centre-Area Lemma, and Its Sharpness

What this chapter assumes

The following facts are quoted from earlier chapters in the form those chapters left them in; none of them is reproved here.

- **(F1) Centre separation** (Chapter 1, Lemma 1.32). Two unit squares with disjoint interiors have centres at distance at least 1.
- **(F2) Branch sufficiency** (Chapter 5, Theorem 5.9). If $\langle Q-P, n\rangle \geq w_F(n) + w_G(n)$ for some unit vector n, then $\operatorname{int} Q(F,P) \cap \operatorname{int} Q(G,Q) = \varnothing$.
- **(F3) Owned separating branch** (Chapter 5, Lemma 5.14). Two unit squares with disjoint interiors admit an oriented separating branch whose unit normal is, up to sign, an axis of one of the two frames. A frame supplying that axis is *an* owner of the branch (Definition 5.16); when both do, one is designated the owner, once and for all.
- **(F4) Threshold of an owned branch** (Chapter 5, Lemma 5.22). For an owned branch between frames F and G,

$$H(F,G;n) = H(t) = \frac{1+\cos t + \sin t}{2} \;\geq\; 1, \qquad t = d_4(F,G) \in [0,\pi/4],$$

  independently of which of the two frames supplies the axis. The excess is $E(t) = H(t) - 1$ (Proposition 5.24).
- **(F5) Subcritical triangle lemma** (Chapter 2, Theorem 2.21). A non-obtuse triangle with all sides at least 1 has all altitudes at least $1/\sqrt{2}$; if in addition its doubled area is less than 1, all sides are less than $\sqrt{2}$ and the triangle is strictly acute.
- **(F6) Existence lemma** (Chapter 9, Theorem 9.16). With the frozen data of an assumed counterexample, the feasible set $X \subseteq \mathbb{R}^4$ is nonempty, closed and bounded, the minimum of D over X is attained, and every minimiser satisfies $0 < D \leq D_0 < 1$ and has a strictly acute centre triangle.
- **(F7) Complete contact** (Chapter 9, Theorem 9.26). At a minimiser, $\langle d_i, n_i\rangle = H_i$ for $i = 1,2,3$.
- **(F8) Multipliers and Euler's identity** (Chapter 10, Theorem 10.6 and Proposition 10.11). There exist $\lambda_1, \lambda_2, \lambda_3 \geq 0$ with $g = \sum_i \lambda_i a_i$; the forces $f_i = \lambda_i n_i$ satisfy the reciprocal equations

$$J_\sigma d_1 = f_2 - f_3, \qquad J_\sigma d_2 = f_3 - f_1, \qquad J_\sigma d_3 = f_1 - f_2;$$

  and $2D = \lambda_1 H_1 + \lambda_2 H_2 + \lambda_3 H_3$.

- **(F9) The sine system** (Chapter 10, Theorem 10.16). With $s_1 = \sigma \det(n_1, n_2)$, $s_2 = \sigma \det(n_2, n_3)$, $s_3 = \sigma \det(n_3, n_1)$,
$$H_1 = \lambda_2 s_1 + \lambda_3 s_3, \qquad H_2 = \lambda_3 s_2 + \lambda_1 s_1, \qquad H_3 = \lambda_1 s_3 + \lambda_2 s_2.$$
- **(F10) Nonpositive sine** (Chapter 10, Proposition 10.26). If $s_i \le 0$ for some i, then $\lambda_1 + \lambda_2 + \lambda_3 \ge 2$ and hence $\lambda_1 H_1 + \lambda_2 H_2 + \lambda_3 H_3 \ge 2$.
- **(F11) Cyclic case** (Chapter 11, Proposition 11.26). If all $s_i > 0$ and P_i owns branch i for each i, then $\lambda_1 + \lambda_2 + \lambda_3 \ge 2$ and hence $\lambda_1 H_1 + \lambda_2 H_2 + \lambda_3 H_3 \ge 2$. The proof is the chain Lemma 11.1 → Proposition 11.5 → Lemma 11.8 → Lemma 11.12 → Proposition 11.16 → Theorem 11.17, and the hypotheses those six results invoke are exactly (H3) contact, (H4) the sine system with $\lambda_i \ge 0$, (H5) $s_i > 0$ and (H6) cyclic ownership, together with $H_i \ge 1$ from (F4).
- **(F12) Transitive case** (Chapter 12, Proposition 12.18). If all $s_i > 0$ and the owner tournament has source P_1, middle vertex P_2 and sink P_3, then $\lambda_1 H_1 + \lambda_2 H_2 + \lambda_3 H_3 \ge 2$. The proof is the chain Lemma 12.1 → Lemma 12.3 → Lemma 12.5 → Lemma 12.10 → Lemma 12.14 → Corollary 12.15, and again each of these invokes only ownership, the frame data, the sine system with $\lambda_i \ge 0$, $H_i \ge 1$ and $s_i > 0$; neither minimality nor D_0 is invoked.
- **(F13) Two geometries** (Chapter 10, Corollary 10.31, resting on Lemma 10.29). There are eight owner assignments. Under the S_3 action on tournaments on the labelled vertex set $\{P_1, P_2, P_3\}$ they form exactly two orbits: the two directed 3-cycles, an orbit of size 2 with stabiliser A_3; and the six transitive tournaments, an orbit of size 6 with trivial stabiliser.

One further item is a construction rather than a fact: the S_3 relabelling of Chapter 10 (Definition 10.20, Lemma 10.21, Proposition 10.22). Chapter 1 supplies three explicit configurations, quoted rather than rebuilt in §§13.5–13.7: Proposition 1.36 (the constant is attained), Example 1.37 (the obtuse degeneration) and Proposition 1.35 (the all-angle bound).

Standing hypotheses

In force from §13.1 to the end of §13.3. §13.4 is commentary on that stretch and refers to the same data, except in Worked Example 13.A, which suspends the hypotheses. A counterexample to Theorem 1.1 is fixed: unit squares $S_i = Q(F_i, P_i)$, $i = 1, 2, 3$, with pairwise disjoint interiors and non-obtuse centre triangle of area less than $\frac{1}{2}$. Its doubled area is $D_0 < 1$. The orientation sign σ, the frames F_i, the normals n_i, the thresholds H_i and the chosen owners are **frozen constants** computed from it. The configuration $x = (P_2, P_3) \in \mathbb{R}^4$, normalised by $P_1 = 0$, ranges over the feasible set; x_* is one fixed minimiser, and D, d_i, λ_i, s_i denote its data. Indices on $P, d, n, H, s, \lambda, f$ are read modulo 3 in $\{1, 2, 3\}$.

Standing warning. "Active branch" means equality in a chosen frozen linear inequality, never geometric tangency of squares; no square is ever moved or rotated.

From §13.5 onwards the standing hypotheses are not in force: those sections work with explicit configurations and are self-contained.

13.1 The chain, restated

The argument of the preceding chapters is summarised below as seven links, labelled (C1)–(C7). Nothing here is proved; each link is a statement and a citation.

(C1) The counterexample. Assume Theorem 1.1 fails: there are admissible unit squares S_1, S_2, S_3 whose centre triangle is non-obtuse of area less than $\frac{1}{2}$. By (F1) the centres are pairwise at distance at least 1, hence distinct; and three distinct collinear points give a straight angle at the middle one, so non-obtuseness makes them noncollinear (Lemma 1.28). Hence $\det(P_2 - P_1, P_3 - P_1) \neq 0$, the sign σ is well defined, and $0 < D_0 < 1$ (Lemma 9.1).

(C2) The freeze. For each of the three pairs, (F3) supplies an owned separating branch. Orient the three normals cyclically, so that branch i reads $\langle d_i, n_i\rangle \geq H_i$ with $d_i = P_{i+1} - P_i$, and choose one owner per branch. By (F4) each $H_i \geq 1$. The frames, normals, thresholds, owners and σ are now constants (Definition 9.4).

(C3) The relaxation. Minimise $D = \sigma \det(P_2 - P_1, P_3 - P_1)$ over centre triples subject to the three frozen branch inequalities, the three non-obtuseness inequalities $\kappa_j \geq 0$, and $0 \leq D \leq D_0$. Normalise $P_1 = 0$, so the configuration is one point $x = (P_2, P_3) \in \mathbb{R}^4$ and each branch function b_i is linear rather than merely affine (Definition 9.14; the linearity is established in Chapter 6).

(C4) The minimiser. By (F6) a minimiser x_* exists, has a strictly acute centre triangle, and satisfies $0 < D \leq D_0 < 1$.

(C5) Complete contact. By (F7) all three frozen branches are active at x_*: $\langle d_i, n_i\rangle = H_i$ for $i = 1, 2, 3$.

(C6) Multipliers and Euler. By (F8) there are $\lambda_1, \lambda_2, \lambda_3 \geq 0$ with $g = \sum_i \lambda_i a_i$, the forces $f_i = \lambda_i n_i$ satisfy the reciprocal equations, and

$$2D \;=\; \lambda_1 H_1 + \lambda_2 H_2 + \lambda_3 H_3.$$

(C7) Scalars. Dotting the reciprocal equations with the normals gives the sine system (F9). Everything that remains is one scalar inequality.

The remaining task is therefore the single inequality $\lambda_1 H_1 + \lambda_2 H_2 + \lambda_3 H_3 \geq 2$, proved in §13.2 by a three-case analysis in which the cases are separated by the signs of the oriented sines and by a tournament on three vertices.

13.2 Exhaustiveness of the three cases

This section isolates the finite tuple of data that the three case propositions use, and shows that relabelling that tuple reduces the analysis to three cases.

13.2.1 Contact data

Relabelling the vertices of a minimisation problem moves the feasible set, the objective and the constraint list; relabelling a finite tuple of vectors and numbers does not. The following definition records what (F10), (F11) and (F12) use.

Definition 13.1 (Contact datum)**.** A *contact datum* is a tuple

$$\Delta = \big(\sigma;\ P_1, P_2, P_3;\ F_1, F_2, F_3;\ n_1, n_2, n_3;\ H_1, H_2, H_3;\ o_1, o_2, o_3;\ \lambda_1, \lambda_2, \lambda_3\big)$$

in which

(a) $\sigma \in \{-1, +1\}$ and P_1, P_2, P_3 are points of $\mathbb{R}^2$ with $D(\Delta) := \sigma \det(P_2 - P_1, P_3 - P_1) > 0$; we write $d_i = P_{i+1} - P_i$, so $d_1 + d_2 + d_3 = 0$;

(b) F_1, F_2, F_3 are frames, and for each i the *owner index* o_i lies in $\{i, i+1\}$ and n_i is a unit vector which is an axis of F_{o_i};

(c) $H_i = H\big(d_4(F_i, F_{i+1})\big)$ for each i, where H is the threshold function of (F4);

(d) $\lambda_i \geq 0$ for each i, and, writing $f_i = \lambda_i n_i$,

$$\textit{(contact)}\quad \langle d_i, n_i\rangle = H_i \qquad (i = 1, 2, 3),$$
$$\textit{(reciprocal)}\quad J_\sigma d_i = f_{i+1} - f_{i+2} \qquad (i = 1, 2, 3).$$

Its *oriented sines* are $s_1 = \sigma \det(n_1, n_2)$, $s_2 = \sigma \det(n_2, n_3)$, $s_3 = \sigma \det(n_3, n_1)$, and its *total* is

$$N(\Delta) \;:=\; \lambda_1 H_1 + \lambda_2 H_2 + \lambda_3 H_3.$$

This is Definition 10.24 of Chapter 10, repeated to name the owner indices o_i and to fix the notation $D(\Delta)$, $N(\Delta)$ used in the orbit argument below.

The following are immediate.

- Every $H_i \geq 1$, by (F4), since $H \geq 1$ on $[0, \pi/4]$.
- The sine system (F9) holds for every contact datum. Its derivation in Chapter 10 used only the reciprocal equations, the two identities $J_\sigma^2 = -I$ and $\langle J_\sigma u, v\rangle = \sigma \det(u, v)$, and the contact equalities — all of which are items (d) of Definition 13.1.
- Consequently $2D(\Delta) = N(\Delta)$, by Corollary 10.17, whose proof likewise uses only the sine system and Proposition 10.15.

The definition is used through the following two observations.

(i) Links (C1)–(C6) produce a contact datum. At the minimiser x_*, take σ, the frames, the frozen normals, thresholds and chosen owners, the centres of x_*, and the multipliers of (F8). Condition (a) holds because $D > 0$ at a minimiser, by (F6); (b) because each frozen branch is owned, by (F3) and the choice made at the freeze; (c) because the threshold of an owned branch is $H(d_4)$ of the two frames involved, by (F4); and (d) is complete contact (F7) together with the reciprocal equations (F8).

(ii) (F10), (F11) and (F12) are statements about contact data. The wording of the earlier chapters differs from what is needed here, so the point is spelled out. Each of the three propositions is stated there under a list of standing hypotheses which includes an assumed counterexample, a minimiser, and $0 < D \leq D_0 < 1$, and a general contact datum satisfies none of those. None of the three proofs uses them: the (F)-list above records, for each proposition, the numbered results its proof consists of and the hypotheses those results invoke. Collecting that information:

- **(F10)** uses the sine system, $\lambda_i \geq 0$, $H_i \geq 1$ and $|s_j| \leq 1$; the last is automatic for unit vectors. All of these are items (c) and (d) of Definition 13.1 together with the first two of the three observations above.
- **(F11)** uses in addition that all $s_i > 0$ and that n_i is an axis of F_i for each i, which is cyclic ownership, item (b) with $o_i = i$. The thresholds enter only through the two-normal formula (Corollary 5.23), applied to the axis n_i of F_i and the axis n_{i+1} of F_{i+1}, and that formula is an identity about two frames and two of their axes. The six results listed under (F11) invoke, between them, hypotheses (H3)–(H6) of Chapter 11 and nothing else.
- **(F12)** uses that n_1 and n_3 are axes of the single frame F_1 — item (b) with $o_1 = 1$ and $o_3 = 1$ — together with $s_i > 0$, the sine system, and the frame distances underlying the thresholds. The six results listed under (F12) invoke nothing beyond those.

In summary: the three propositions are proved from hypotheses that a contact datum supplies, and are quoted here under those hypotheses. The standing lists of Chapters 10–12 are wider than their proofs require. The case propositions are thus statements about tuples, and tuples may be relabelled.

13.2.2 The relabelling lemma

Lemma 13.2 (Relabelling). *Let Δ be a contact datum and let $\pi \in S_3$. Define Δ^π as follows: $P'_i = P_{\pi(i)}$ and $F'_i = F_{\pi(i)}$; $\sigma' = \operatorname{sign}(\pi)\,\sigma$; branch i of Δ^π is the branch of Δ joining $P_{\pi(i)}$ and $P_{\pi(i+1)}$, with the same threshold H'_i, the same owner (relabelled, so $o'_i = \pi^{-1}(o)$ where o was the old owner index), the same multiplier λ'_i, and with normal $n'_i = \pm n$, the sign being $-$ exactly when the relabelling reverses that branch's ordered pair. Then Δ^π is a contact datum, and*

$$D(\Delta^\pi) = D(\Delta), \qquad N(\Delta^\pi) = N(\Delta),$$

while (s'_1, s'_2, s'_3) is a permutation of (s_1, s_2, s_3). Moreover $\Delta^{\pi\rho} = (\Delta^\pi)^\rho$ for $\pi, \rho \in S_3$ and $\Delta^{\mathrm{id}} = \Delta$, so this is an action of S_3 on contact data.

The order in $\Delta^{\pi\rho} = (\Delta^\pi)^\rho$ makes this a right action, as an operation defined by substituting into indices must be. The action on tournaments in §13.2.3 is written on the left, in the usual way; the inversion $\pi \mapsto \pi^{-1}$ that converts one convention into the other is what produces the π^{-1} in Lemma 13.4.

Proof. The group S_3 is generated by the cyclic shift $\rho\colon i \mapsto i+1$ and the transposition $\tau = (2\ 3)$: the subgroup they generate contains an element of order 3 and an element of order 2, so its order is divisible by 6. Since relabelling by a product is the composite of the relabellings — immediate from the description, which reads off each new symbol from the vertex order alone — it suffices to verify the two generators.

The cyclic shift. Here $P_1' = P_2$, $P_2' = P_3$, $P_3' = P_1$, so $d_i' = P_{i+2} - P_{i+1} = d_{i+1}$, and $\sigma' = \sigma$ because $\operatorname{sign}(\rho) = +1$; indeed $\det(P_2 - P_1, P_3 - P_1)$ is unchanged by a cyclic permutation of the three vertices, so the new determinant has the old sign and

$$D(\Delta^\rho) = \sigma' \det(P_3 - P_2, P_1 - P_2) = \sigma \det(P_2 - P_1, P_3 - P_1) = D(\Delta) > 0.$$

New branch i joins the ordered pair (P_{i+1}, P_{i+2}), which is old branch $i+1$ in its original order; no pair is reversed, so

$$n_i' = n_{i+1}, \qquad H_i' = H_{i+1}, \qquad \lambda_i' = \lambda_{i+1}, \qquad F_i' = F_{i+1}.$$

Now check the four conditions. (a) is done. (b): $n_i' = n_{i+1}$ is an axis of $F_{o_{i+1}}$, and $o_{i+1} \in \{i+1, i+2\}$, which after relabelling is $\{i, i+1\}$ as required. (c): $H\big(d_4(F_i', F_{i+1}')\big) = H\big(d_4(F_{i+1}, F_{i+2})\big) = H_{i+1} = H_i'$. (d): contact reads $\langle d_i', n_i'\rangle = \langle d_{i+1}, n_{i+1}\rangle = H_{i+1} = H_i'$, and, with $f_i' = \lambda_{i+1} n_{i+1} = f_{i+1}$,

$$J_{\sigma'} d_i' = J_\sigma d_{i+1} = f_{i+2} - f_{i+3} = f_{i+1}' - f_{i+2}',$$

the middle equality being the reciprocal equation of Δ with index $i{+}1$. Finally $s_i' = \sigma \det(n_{i+1}, n_{i+2}) = s_{i+1}$, so $(s_1', s_2', s_3') = (s_2, s_3, s_1)$, and $N(\Delta^\rho) = \sum_i \lambda_{i+1} H_{i+1} = N(\Delta)$.

The transposition $\tau = (2\ 3)$. Here $P_1' = P_1$, $P_2' = P_3$, $P_3' = P_2$, and $\sigma' = -\sigma$. Compute the edges directly:

$$d_1' = P_3 - P_1 = -d_3, \qquad d_2' = P_2 - P_3 = -d_2, \qquad d_3' = P_1 - P_2 = -d_1.$$

Every pair is reversed. New branch 1 joins (P_1, P_3), which is old branch 3 reversed; new branch 2 joins (P_3, P_2), old branch 2 reversed; new branch 3 joins (P_2, P_1), old branch 1 reversed. Hence

$$n_1' = -n_3, \quad n_2' = -n_2, \quad n_3' = -n_1;$$
$$(H_1', H_2', H_3') = (H_3, H_2, H_1), \qquad (\lambda_1', \lambda_2', \lambda_3') = (\lambda_3, \lambda_2, \lambda_1),$$

and the owners are unchanged as squares, since a square carries its label with it.

(a): $D(\Delta^\tau) = \sigma' \det(P_3 - P_1, P_2 - P_1) = (-\sigma)\big(-\det(P_2 - P_1, P_3 - P_1)\big) = D(\Delta) > 0$. Both sign changes occur and they cancel.

(b): $-n_3$ is an axis of a frame whenever n_3 is, because the axes of a frame come in pairs $\pm e, \pm f$; and the owner index is transported correctly, since o_1' is the label of the old owner of old branch 3 after applying $\tau^{-1} = \tau$, and old branch 3 joins P_3, P_1 whose new labels are $2, 1$, so $o_1' \in \{1, 2\}$ as required. The same check applies to the other two branches.

(c): $H_1' = H_3 = H\big(d_4(F_3, F_1)\big) = H\big(d_4(F_1', F_2')\big)$ since $F_1' = F_1$, $F_2' = F_3$ and d_4 is symmetric; similarly $H_2' = H_2 = H(d_4(F_3, F_2))$ and $H_3' = H_1 = H(d_4(F_2, F_1))$.

(d) *Contact.* For each i there is a j with $d_i' = -d_j$, $n_i' = -n_j$, $H_i' = H_j$, so

$$\langle d_i', n_i'\rangle = \langle -d_j, -n_j\rangle = \langle d_j, n_j\rangle = H_j = H_i'.$$

The sign flips cancel in pairs; the remaining verifications are instances of the same cancellation.

(d) *Reciprocal.* Since $\sigma' = -\sigma$ we have $J_{\sigma'} = -J_\sigma$, and the new forces are $f_1' = \lambda_3(-n_3) = -f_3$, $f_2' = -f_2$, $f_3' = -f_1$. Then

$$J_{\sigma'} d_1' = (-J_\sigma)(-d_3) = J_\sigma d_3 = f_1 - f_2 \quad \text{and} \quad f_2' - f_3' = (-f_2) - (-f_1) = f_1 - f_2,$$

so the first new equation holds; likewise $J_{\sigma'} d_2' = J_\sigma d_2 = f_3 - f_1$ and $f_3' - f_1' = (-f_1) - (-f_3) = f_3 - f_1$, and $J_{\sigma'} d_3' = J_\sigma d_1 = f_2 - f_3$ and $f_1' - f_2' = (-f_3) - (-f_2) = f_2 - f_3$. Three sign flips — of σ, of the edge and of the forces — cancel each time.

Sines and totals. Here the cancellation is displayed in full:

$$\begin{aligned} s_1' &= \sigma' \det(n_1', n_2') = (-\sigma)\det(-n_3, -n_2) \\ &= (-\sigma)\det(n_3, n_2) = \sigma \det(n_2, n_3) = s_2, \end{aligned}$$

the two sign flips of the normals cancelling in the bilinear determinant, and the flip of σ being absorbed by the antisymmetry of det. In the same way $s_2' = s_1$ and $s_3' = s_3$. Finally $N(\Delta^\tau) = \lambda_3 H_3 + \lambda_2 H_2 + \lambda_1 H_1 = N(\Delta)$, a rearrangement of the same three products.

The action property is the observation already used: relabelling reads every new symbol off the new vertex order, so relabelling by π and then by ρ is relabelling by $\pi\rho$, and relabelling by the identity does nothing. □

Remark. Nothing about the minimisation problem was used: not the feasible set, not minimality, not D_0. This is what licenses the "without loss of generality" of §13.2.4: the relabelled datum Δ^π need not arise from any relabelled minimisation problem. It is a tuple satisfying the hypotheses of (F10)–(F12), and those propositions ask for nothing more.[1]

Relation to Chapter 10. That chapter proved the same computation (Lemma 10.21 and Proposition 10.22) for the data of the minimiser, with the standing hypotheses in force. Lemma 13.2 is that statement with the standing hypotheses removed; the verification is unchanged.

13.2.3 The owner tournament under relabelling

Definition 13.3 (Owner tournament)**.** The *owner tournament* $\mathcal{T}(\Delta)$ of a contact datum Δ is the tournament on the labelled vertex set $\{P_1, P_2, P_3\}$ whose edge for branch i joins P_i and P_{i+1} and is directed away from its owner P_{o_i}, towards the other endpoint.

This is Definition 10.30 with the standing hypotheses of that chapter removed. As in Chapter 5, owner arrows are drawn from the owner to the other endpoint and never along a normal, and branch normals n_i are drawn askew to the edges d_i, being axes of frozen frames rather than functions of the edges.

The orbit count is (F13). What it does not supply is how the relabelling of §13.2.2, an operation on tuples with no minimisation problem behind it, acts on tournaments; that is the following lemma.

Lemma 13.4 (Relabelling acts on owner tournaments)**.** *Let S_3 act on tournaments on the labelled vertex set $\{P_1, P_2, P_3\}$ on the left, by declaring $\alpha \cdot \mathcal{T}$ to have the edge $\alpha(a) \to \alpha(b)$ whenever $\mathcal{T}$ has the edge $a \to b$. Then for every contact datum Δ and every $\pi \in S_3$,*

$$\mathcal{T}(\Delta^\pi) \;=\; \pi^{-1} \cdot \mathcal{T}(\Delta).$$

Consequently every contact datum may be relabelled so that its owner tournament is one of

(a) cyclic*: $o_i = i$ for every i, that is, P_i owns branch i;*

(b) transitive*: $o_1 = 1$, $o_2 = 2$, $o_3 = 1$, that is, P_1 is the source, P_2 the middle vertex and P_3 the sink.*

Proof. That $\alpha \cdot \mathcal{T}$ is again a tournament, and that $(\alpha\beta) \cdot \mathcal{T} = \alpha \cdot (\beta \cdot \mathcal{T})$, are immediate from the definition. Owner assignments and tournaments correspond bijectively, the owner being recovered as the tail of its edge; so $\mathcal{T}$ loses no information.

For the displayed identity, let $\mathcal{T} = \mathcal{T}(\Delta)$ and let Δ^π be as in Lemma 13.2. Branch i of Δ^π is the old branch joining $P_{\pi(i)}$ and $P_{\pi(i+1)}$, and it is directed away from the same square, since a square carries its label with it and the relabelling only renames the labels. So $\mathcal{T}(\Delta^\pi)$ has the edge $i \to i+1$ exactly when $\mathcal{T}$ has the edge $\pi(i) \to \pi(i+1)$, which is the assertion $\mathcal{T}(\Delta^\pi) = \pi^{-1} \cdot \mathcal{T}(\Delta)$.

For the consequence: since $\pi \mapsto \pi^{-1}$ is a bijection of S_3, the tournaments $\mathcal{T}(\Delta^\pi)$, as π ranges over S_3, are exactly the members of the orbit of $\mathcal{T}(\Delta)$. By (F13) there are exactly two orbits,

[1] As set out in Chapter 10, a "without loss of generality" requires a group, a proof that hypotheses and conclusion are invariant, and an identification of orbits. Here the group is S_3, the invariance is Lemma 13.2, the orbits are counted in (F13), and Lemma 13.4 connects the two.

the two directed 3-cycles and the six transitive tournaments. It remains only to see that (a) and (b) are representatives, one from each. For (a), $o_i = i$ directs edge i from P_i to P_{i+1}, giving $P_1 \to P_2 \to P_3 \to P_1$, a directed 3-cycle; and P_i owns branch i, which is the normal form of Chapter 11. For (b), $o_1 = 1$ gives $P_1 \to P_2$; $o_2 = 2$ gives $P_2 \to P_3$; and $o_3 = 1$ gives $P_1 \to P_3$, since branch 3 joins P_3 and P_1 and is owned by P_1. The out-degrees are $2, 1, 0$ at P_1, P_2, P_3, so this tournament is transitive with P_1 the source, P_2 the middle vertex and P_3 the sink in that order, the normal form of Chapter 12. ☐

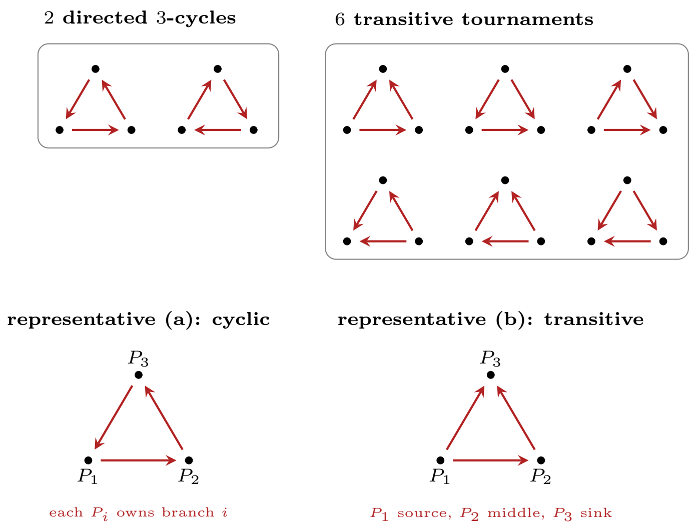


Figure 13.1: The eight owner assignments, drawn as tournaments with red owner arrows pointing from owner to the other endpoint. They fall into an orbit of size 2 (the directed 3-cycles) and an orbit of size 6 (the transitive tournaments), by (F13). Below, the two representatives that Lemma 13.4 makes reachable, used by Chapters 11 and 12. In every panel the vertices are P_1 (lower left), P_2 (lower right), P_3 (top).

A branch may have two legitimate owners. If the normal n_i is an axis of both F_i and F_{i+1}, for instance whenever the two frames are equal, then either square may be declared the owner. The freeze chose one, once (Definition 5.16), and the owner tournament is a function of that choice, so two different choices can send the same normals and thresholds into different cases of the analysis. The conclusion is the same either way. The owner indices o_i occur in neither $D(\Delta)$ nor $N(\Delta)$ nor the oriented sines, all of which are functions of σ, the centres, the normals, the thresholds and the multipliers alone; so every admissible choice of owners yields the same two numbers $D(\Delta)$ and $N(\Delta)$, and Proposition 13.5 bounds that number below by 2 along whichever route the choice selects. Worked Example 13.A gives a configuration in which all three frames coincide, so that all eight owner assignments are legitimate and all eight give $N = 2$; there $s_2 = 0$, so all eight are settled by (F10) before any tournament is formed, and the example does not exhibit the tournament branching.

Ownership is not the direction of the normal. The owner arrow of branch i runs between P_i and P_{i+1}; the normal n_i is an axis of a frozen frame and points wherever that frame points. The two are independent and should not be drawn together.

13.2.4 The exhaustiveness proposition

Proposition 13.5 (Exhaustiveness)**.** *For every contact datum Δ,*

$$N(\Delta) = \lambda_1 H_1 + \lambda_2 H_2 + \lambda_3 H_3 \;\geq\; 2.$$

Proof. Either some $s_i \leq 0$, or all three are positive; these are the only possibilities, and they are mutually exclusive.

Case 1: $s_i \le 0$ for some i. By observation (ii) of §13.2.1, (F10) applies to Δ and gives $N(\Delta) \ge 2$ directly. (Its proof treats the index $i = 3$ and reduces the other two indices to that one by the cyclic relabelling, which is Lemma 13.2 for ρ and ρ^2; that reduction is internal to the proposition.)

Case 2: $s_1, s_2, s_3 > 0$. Form the owner tournament $\mathcal{T}(\Delta)$. By Lemma 13.4 there is $\pi \in S_3$ for which $\mathcal{T}(\Delta^\pi)$ is representative (a) or representative (b). Replace Δ by Δ^π. By Lemma 13.2, Δ^π is again a contact datum; its sines are a permutation of s_1, s_2, s_3 and so are again all positive; the contact equalities, the reciprocal equations and hence the sine system hold in identical displayed form; and $N(\Delta^\pi) = N(\Delta)$.

If the tournament of Δ^π is representative (a), then P_i owns branch i for each i, so (F11) applies and gives $N(\Delta^\pi) \ge 2$. If it is representative (b), then P_1, P_2, P_3 are the source, middle vertex and sink in that order, so (F12) applies and gives $N(\Delta^\pi) \ge 2$. Either way $N(\Delta) = N(\Delta^\pi) \ge 2$. □

The proof does not need to know which of the two directed 3-cycles occurs, nor which of the six transitive tournaments; both possibilities are absorbed by the orbit count. The two cyclic tournaments are exchanged by an odd permutation, so reaching representative (a) may require flipping σ. This is why Lemma 13.2 was proved for τ as well as ρ, and why $\sigma = +1$ is nowhere assumed.

13.3 Proof of Theorem 1.1

We restate the theorem in the form of Chapter 1.

> **Theorem 1.1 (Centre-area lemma).** *Let S_1, S_2, S_3 be unit squares in the plane, rotated arbitrarily and independently, with pairwise disjoint interiors, and let P_1, P_2, P_3 be their centres. If the triangle $P_1P_2P_3$ is non-obtuse, then*
>
> $$\mathrm{Area}(P_1P_2P_3) \;\ge\; \tfrac{1}{2}.$$

Proof. Suppose not. Fix admissible unit squares S_1, S_2, S_3 whose centre triangle is non-obtuse of area less than $\frac{1}{2}$, and let $D_0 < 1$ be its doubled area; by (C1), $D_0 > 0$ and σ is defined. Freeze one owned branch per pair, with a chosen owner, as in (C2), and form the relaxed minimisation problem (C3).

By (F6) a minimiser x_* exists, has a strictly acute centre triangle, and satisfies $0 < D \le D_0 < 1$. By (F7) all three frozen branches are active at x_*, and by (F8) there are multipliers $\lambda_1, \lambda_2, \lambda_3 \ge 0$ with

$$2D \;=\; \lambda_1 H_1 + \lambda_2 H_2 + \lambda_3 H_3.$$

By observation (i) of §13.2.1, the data of x_* form a contact datum, so Proposition 13.5 gives $\lambda_1 H_1 + \lambda_2 H_2 + \lambda_3 H_3 \ge 2$, whence $2D \ge 2$, that is

$$D \;\ge\; 1.$$

Both $D \ge 1$ and $D \le D_0 < 1$ assert something about the doubled area of the same configuration, namely the minimiser x_*, and they are incompatible. Hence no counterexample exists: every admissible triple whose centre triangle is non-obtuse satisfies $\mathrm{Area}(P_1P_2P_3) \ge \frac{1}{2}$. □

Collinear triples. Three collinear centres span a degenerate "triangle" of area 0, which is less than $\frac{1}{2}$, but such a triple is not non-obtuse. If P_1, P_2, P_3 are distinct and collinear, one of them lies strictly between the other two, and at that vertex the two edge vectors are nonzero multiples of a common unit vector with coefficients of opposite sign, so their inner product is negative and the angle there is π — obtuse (Lemma 1.28). If two of them coincide the angles are undefined, and (F1) forbids it. So the hypothesis of Theorem 1.1 is vacuous on collinear triples, and the theorem says nothing about them. What holds for them without any hypothesis is the all-angle bound (Proposition 1.35, recalled in §13.7), whose right-hand side is 0 there.

13.4 Which configuration each inequality refers to

The proof of §13.3 quotes results about two different configurations. This section records which statement is about which.

13.4.1 Two configurations, one symbol table

symbol	belongs to	status
D_0	the assumed counterexample	a fixed real number, $0 < D_0 < 1$
σ, F_i, n_i, H_i, owners	computed once, at the counterexample	frozen constants; never recomputed
$x = (P_2, P_3)$, d_i, D	a point of the feasible set X	varying, until (C4)
x_*, and D, d_i, λ_i, s_i at x_*	the minimiser	fixed once and for all after (C4)
$Q(F_i, P_i^*)$	the squares at the minimiser	admissible, but not the original squares

The inequality $D \geq 1$ proved in §13.3 is about the minimiser, not about the counterexample. The counterexample enters the final step only through the frozen data and through the single number D_0.

13.4.2 What the relaxation is for, and in which direction it is used

The relaxation admits two readings, and only one of them is used below.

Sufficiency reading. By (F2), every feasible point of the relaxed problem yields three admissible unit squares: the frames are frozen, and a branch certifies disjointness of a pair from one-dimensional projection data of those frames alone (Lemma 9.9). So a lower bound proved over the relaxation is a lower bound over a class of configurations containing the admissible ones with those frames and those branches. This is what a direct phrasing of the argument, rather than one by contradiction, would use.

What the contradiction needs. Two things: that the counterexample's own centre triple is feasible, so that the feasible set is nonempty; and that minimality then forces $D \leq D_0$ at the minimiser. Sufficiency is not logically required for the contradiction. Chapter 9 nevertheless records it, since it is what makes the relaxation the right object to minimise over.

13.4.3 Why there is no circularity

1. **Theorem 1.1 is never used inside the argument.** Every ingredient (F1)–(F13) is proved without it. The dependency chain is: Chapters 2 and 5 use only elementary plane geometry and the convexity of Chapter 4; Chapters 6 to 8 use only analysis and convexity; Chapters 9 to 12 use only the frozen data and the scalar toolkit of Chapter 3.

2. **All choices are made after the counterexample is fixed, and there are finitely many of them.** One branch per pair, drawn from the finite candidate list supplied by the separating axis theorem of Chapter 5; one owner per branch; one orientation sign; one minimiser; one relabelling. None of these choices depends on the conclusion.

3. **The two bounds bound the same number**, namely D at x_*. In §13.3 the number D_0 enters in exactly two roles: as the cap in the feasibility constraint $D \leq D_0$, and in the nonemptiness check that the counterexample itself is feasible.

4. **The minimiser is not claimed to be geometrically meaningful.** The squares $Q(F_i, P_i^*)$ are admissible, by (F2), but they need not be the original squares, need not touch, and nothing is asserted about them beyond feasibility.

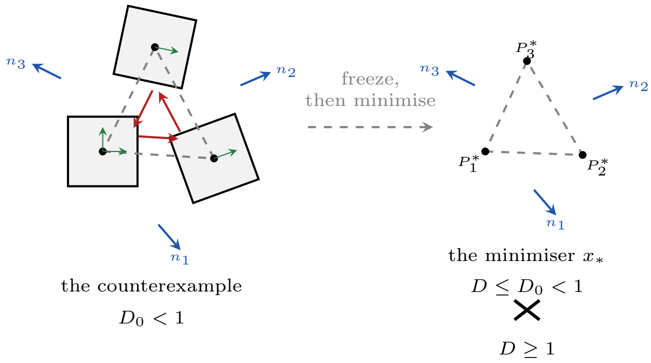


Figure 13.2: The two configurations of §13.4. *Left*: the assumed counterexample, with frames, branch normals drawn askew to the edges, and owner arrows drawn between centres rather than along a normal. *Right*: the minimiser, which carries the same frozen normals but different centres, and at which the two incompatible bounds are derived; the black cross marks the contradiction. The dashed arrow transports data — frames, normals, thresholds, owners, σ and D_0 — not squares.

Worked Example 13.A: the chain evaluated at the equality configuration

The standing hypotheses are suspended for this example. The configuration below is neither a counterexample nor a minimiser, and no minimisation problem is in play; what is computed is the list of objects the chain produces, evaluated on one explicit triple of squares. Take axis-parallel unit squares, all with the standard frame $F = (e_1, e_2)$, centred at

$$P_1 = (0,0), \qquad P_2 = (1,0), \qquad P_3 = (0,1).$$

Orientation and area. $\det(P_2 - P_1, P_3 - P_1) = \det\big((1,0),(0,1)\big) = 1 > 0$, so $\sigma = +1$ and $D = 1$. For this example only we use coordinates for the quarter turn: $J_{+1}(a,b) = (-b,a)$.

Edges. $d_1 = (1,0)$, $d_2 = P_3 - P_2 = (-1,1)$, $d_3 = P_1 - P_3 = (0,-1)$; and $d_1 + d_2 + d_3 = 0$.

Thresholds. All three frames coincide, so all three frame distances are 0 and

$$H_1 = H_2 = H_3 = H(0) = \frac{1 + \cos 0 + \sin 0}{2} = 1.$$

Branches. Take

$$n_1 = (1,0), \qquad n_2 = (0,1), \qquad n_3 = (0,-1),$$

each an axis of the common frame, so each branch is owned, by either of its two squares; the choice is made arbitrarily. Then

$$\langle d_1, n_1\rangle = 1, \qquad \langle d_2, n_2\rangle = \langle(-1,1),(0,1)\rangle = 1,$$

$$\langle d_3, n_3\rangle = \langle(0,-1),(0,-1)\rangle = 1,$$

so all three branches are active: complete contact holds.

Oriented sines.

$$s_1 = \det\big((1,0),(0,1)\big) = 1, \qquad s_2 = \det\big((0,1),(0,-1)\big) = 0,$$

$$s_3 = \det\big((0,-1),(1,0)\big) = 1.$$

So $s_2 = 0$, and the sharp configuration falls under the nonpositive-sine case rather than either tournament case.

Multipliers. With $P_1 = 0$ the configuration point is $x = \big((1,0),(0,1)\big)$, the objective gradient is $g = (-J_{+1}P_3,\ J_{+1}P_2) = \big((1,0),(0,1)\big)$, and the constraint gradients are

$$a_1 = \big((1,0),(0,0)\big), \qquad a_2 = \big((0,-1),(0,1)\big), \qquad a_3 = \big((0,0),(0,1)\big).$$

Reading $g = \sum_i \lambda_i a_i$ block by block: the first block gives $(1,0) = \lambda_1(1,0) + \lambda_2(0,-1)$, so $\lambda_1 = 1$ and $\lambda_2 = 0$; the second gives $(0,1) = \lambda_2(0,1) + \lambda_3(0,1)$, so $\lambda_3 = 1$. Thus $(\lambda_1, \lambda_2, \lambda_3) = (1,0,1)$ and the forces are

$$f_1 = (1,0), \qquad f_2 = (0,0), \qquad f_3 = (0,-1).$$

The three identities, checked. Sine system:

$$\lambda_2 s_1 + \lambda_3 s_3 = 0 + 1 = 1 = H_1, \qquad \lambda_3 s_2 + \lambda_1 s_1 = 0 + 1 = 1 = H_2,$$

$$\lambda_1 s_3 + \lambda_2 s_2 = 1 + 0 = 1 = H_3.$$

Reciprocal equations:

$$J_\sigma d_1 = (0,1) = f_2 - f_3, \qquad J_\sigma d_2 = J_\sigma(-1,1) = (-1,-1) = f_3 - f_1,$$
$$J_\sigma d_3 = J_\sigma(0,-1) = (1,0) = f_1 - f_2.$$

Euler: $2D = 2$ and $\lambda_1 H_1 + \lambda_2 H_2 + \lambda_3 H_3 = 1 + 0 + 1 = 2$.

Two legitimate branches for one pair. The branch for the pair (S_2, S_3) admits two owned normals of threshold 1, since $d_2 = (-1,1)$ has both coordinates of absolute value 1: the choice $n_2 = (0,1)$ above, and the choice $n_2 = (-1,0)$, for which $\langle d_2, (-1,0)\rangle = 1 = H_2$ as well. Rerun the computation with $n_2 = (-1,0)$. The sines become

$$s_1 = \det\big((1,0),(-1,0)\big) = 0, \qquad s_2 = \det\big((-1,0),(0,-1)\big) = 1,$$

$$s_3 = \det\big((0,-1),(1,0)\big) = 1,$$

so now $s_1 = 0$. The constraint gradients become $a_1 = \big((1,0),(0,0)\big)$, $a_2 = \big((1,0),(-1,0)\big)$, $a_3 = \big((0,0),(0,1)\big)$, and $g = \sum_i \lambda_i a_i$ gives $\lambda_1 + \lambda_2 = 1$ from the first block and $\lambda_2 = 0$, $\lambda_3 = 1$ from the second; so $(\lambda_1, \lambda_2, \lambda_3) = (1,0,1)$ again, the forces are unchanged, and $N = 2$ again. The ambiguity here is in the branch rather than the owner: one pair of squares admits two owned normals, giving two frozen systems and the same bound. The freeze picks one and the argument runs on it. (The separate ambiguity in the owner, discussed after Lemma 13.4, is not visible here: the oriented sines are computed from the normals alone, so no owner choice changes them.)

Three observations.

(i) *Every quoted inequality is an equality here.* $H_i = 1$ for each i (the threshold bound of (F4) is attained), $N = 2$ (the target inequality is attained), and $D = 1$ (the conclusion of Theorem 1.1 is attained).

(ii) *One force vanishes.* $f_2 = 0$, so a vertex of the force triangle $f_1 f_2 f_3$ sits at the origin of the force plane. The triangle itself is not degenerate: by the reciprocal equations its edges are $J_\sigma d_1, J_\sigma d_2, J_\sigma d_3$, so it is congruent to the centre triangle and has area $\frac{1}{2}$; here its vertices are $(1,0)$, $(0,0)$, $(0,-1)$. What holds is that the origin lies on the boundary of the force triangle rather than strictly inside it, because $\lambda_2 = 0$. The two degenerations are independent: Chapter 10 records that $s_i > 0$ for all i is what puts the three rays $\mathbb{R}_{\geq 0} n_i$ in positive cyclic order, whereas a vanishing multiplier puts the corresponding f_i at the origin. Here both happen at once, but independently: $s_2 = 0$ says that $n_2 = (0,1)$ and $n_3 = (0,-1)$ are antiparallel, and it has nothing to do with λ_2.

(iii) *The configuration is not a counterexample.* Its doubled area is $D = 1$, not $D < 1$, so it is not a feasible point of the relaxed problem and the argument of §13.3 is never run on it; it lies on the boundary of the region the proof excludes. This also explains why the conclusion of (F6) that minimisers are acute is consistent with the extremal triangle being right-angled: acuteness was derived from $D < 1$ via (F5), and here $D = 1$.

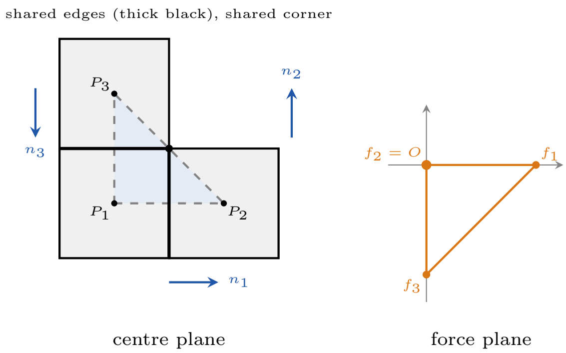


Figure 13.3: The equality configuration of Worked Example 13.A, drawn to scale. The first two squares share the edge on $x = \frac{1}{2}$ and the first and third the edge on $y = \frac{1}{2}$ (thick black); the second and third share only the corner $(\frac{1}{2}, \frac{1}{2})$. The centre triangle (shaded) has area $\frac{1}{2}$. The normals are drawn outside the squares to keep them clear of the edges. *Right*: the force plane. The force triangle is congruent to the centre triangle, as the reciprocal equations require; here $\lambda_2 = 0$, so the vertex f_2 sits at the origin and the origin lies on the boundary of the triangle rather than inside it.

13.5 The constant is attained

Proposition 1.36 of Chapter 1 states that the constant $\frac{1}{2}$ in Theorem 1.1 is attained, by the axis-parallel unit squares centred at $(0,0)$, $(1,0)$, $(0,1)$: their interiors are pairwise disjoint and their centre triangle is right isosceles with legs 1, of area $\frac{1}{2}$.

The three squares here are not pairwise disjoint: two pairs share a whole edge and the third pair shares a corner. The construction fails if one demands disjoint closed squares, and no configuration of three pairwise disjoint closed unit squares attains the constant. The hypothesis of Theorem 1.1 is about interiors, and its sharp case is one in which the closed squares touch, which is why Chapter 1 draws the distinction.

This configuration is the one of Worked Example 13.A, where every inequality in the chain is an equality: each threshold equals its lower bound 1, the target sum equals its bound 2, and the doubled area equals its bound 1. Any improvement to the constant would have to improve one of those three.

13.6 The angle hypothesis cannot be dropped

Example 1.37 of Chapter 1 states the following. For $0 < \varepsilon < \frac{1}{2}$ the axis-parallel unit squares centred at

$$P_1 = (0,0), \qquad P_2 = (1,\varepsilon), \qquad P_3 = (2,0)$$

have pairwise disjoint interiors — their shadows on the horizontal axis are the intervals $[-\frac{1}{2}, \frac{1}{2}]$, $[\frac{1}{2}, \frac{3}{2}]$, $[\frac{3}{2}, \frac{5}{2}]$, whose interiors are pairwise disjoint — and their centre triangle has area ε and is obtuse at the middle vertex. So the infimum of the centre-triangle area over admissible triples, with no angle hypothesis, is 0.

Two further observations are available now. The first is that the verification of admissibility is an instance of the branch idea. All three frames are the standard one, so all three thresholds are $H(0) = 1$; the separating direction is e_1 in each case, and with $d_1 = (1,\varepsilon)$, $d_2 = (1,-\varepsilon)$, $d_3 = (-2,0)$ the three branch inequalities read

$$\langle d_1, e_1 \rangle = 1 \ \geq\ 1, \qquad \langle d_2, e_1 \rangle = 1 \ \geq\ 1, \qquad \langle d_3, -e_1 \rangle = 2 \ \geq\ 1.$$

The certificate is one-dimensional projection data and nothing else, which is (F2).

Where the proof breaks without non-obtuseness

The second is to trace the failure through the chain. Keep the frozen data just displayed.

- **The displayed configuration is not a minimiser.** Its third branch is slack: the value is 2 against a threshold of 1. This is not a failure, since (F7) asserts complete contact at a minimiser and says nothing about an arbitrary feasible point; but the minimisation the chain needs is no longer available, as the next item shows.

- **The feasible set is unbounded and the infimum drops to** 0. Delete the three non-obtuseness constraints $\kappa_j \geq 0$ from Definition 9.14 and keep the three branch inequalities together with $0 \leq D \leq D_0$. Since $d_3 = -(d_1 + d_2)$, the third branch inequality $\langle d_1 + d_2, e_1 \rangle \geq 1$ is implied by the first two, so the collinear configurations

$$P_1 = (0, 0), \qquad P_2 = (R, 0), \qquad P_3 = (2R, 0) \qquad (R \geq 1)$$

are all feasible, with $D = 0$. The feasible set is therefore unbounded, and the infimum of D over it is 0, attained.

- **The remaining steps have nothing to act on.** At those configurations $D = 0$, so the objective's infimum is 0 and no positive lower bound on D survives: the target $D \geq 1$ is false for the relaxed problem. Had the counterexample itself been collinear, σ would not have been defined, since (C1) obtains it as the sign of a determinant that is now zero. Nor is the route to boundedness taken in Chapter 9 available: the subcritical triangle lemma (F5) does not apply, its hypothesis of non-obtuseness having been deleted, so neither the bound $\|d_i\| < \sqrt{2}$ that gave boundedness nor the strict acuteness that made the κ_j slack survives. What fails is not the technique but the conclusion: the family above has areas tending to 0, so no argument can produce a positive lower bound once non-obtuseness is dropped.

Non-obtuseness therefore enters at least two of the three places listed in §13.11: noncollinearity, giving $D > 0$ and a well-defined σ; and the subcritical triangle lemma, giving boundedness and strict acuteness. The example shows that neither use can be removed.

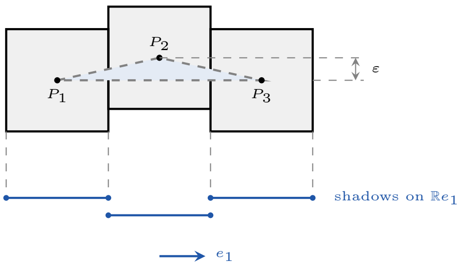


Figure 13.4: The obtuse degeneration of Example 1.37, with ε exaggerated. The three shadows on the axis $\mathbb{R}e_1$ (blue, with grey dashed guides from the square corners) have pairwise disjoint interiors, which certifies admissibility by one-dimensional data alone. The centre triangle (shaded) has area ε and is obtuse at P_2.

13.7 The all-angle bound

Proposition 1.35 of Chapter 1 states the following. If S_1, S_2, S_3 are unit squares with pairwise disjoint interiors and centres P_1, P_2, P_3, then for every angle θ of the centre triangle,

$$\mathrm{Area}(P_1P_2P_3) \;\geq\; \tfrac{1}{2}\sin\theta.$$

The proof there is two lines: (F1) makes the three centres pairwise at distance at least 1, hence distinct, so all three angles are defined; and Area $= \frac{1}{2}ab\sin\theta$ with $a, b \geq 1$ the two sides adjacent to θ. No angle hypothesis is needed, and the weakness of the bound lies in the factor $\sin\theta$, which can be arbitrarily small.

Applying the bound to the largest angle is the strongest instance, and by Example 1.37 that instance is asymptotically sharp: for the degenerating family, $\sin\theta = 2\varepsilon/(1 + \varepsilon^2)$ at the middle vertex while the area is ε, so

$$\frac{\text{Area}}{\frac{1}{2}\sin\theta} = 1+\varepsilon^2 \longrightarrow 1 \qquad (\varepsilon \to 0).$$

The computation is Exercise 1.6(a).

The role of non-obtuseness. Since $\sin\theta \le 1$ always, Theorem 1.1 implies Proposition 1.35 for non-obtuse triangles and strictly improves it except when some angle is right; non-obtuseness removes the factor $\sin\theta$. The degeneration cannot be repaired by excluding small angles: $\sin\theta$ is small at a small angle too, but non-obtuseness does not forbid small angles, it forbids large ones, and it is the large angle that the degeneration exploits. For a non-obtuse triangle every angle is at most $\pi/2$ while the largest is at least $\pi/3$, so the largest angle $\theta_{\max}$ satisfies $\sin\theta_{\max} \ge \sqrt{3}/2$, and Proposition 1.35 already gives Area $\ge \sqrt{3}/4$ there. Exercise 13.5 shows that $\sqrt{3}/4$ is the exact limit of what pairwise information can deliver, and $\sqrt{3}/4 < \frac{1}{2}$.

13.8 No four-centre analogue

Theorem 1.1 is a statement about triples. The natural four-point candidate is the area of the convex hull of four centres, and it admits no positive lower bound.

The vocabulary is that of Chapter 1. Four points q_0, q_1, q_2, q_3 are in *strict convex position in that cyclic order* if the four *turn determinants*

$$\delta_k := \det(q_{k+1}-q_k,\ q_{k+2}-q_{k+1}) \qquad (k=0,1,2,3,\ \text{indices modulo } 4)$$

are all strictly positive; the *quadrilateral spanned* by such a list is the union of the two triangles $q_0q_1q_2$ and $q_0q_2q_3$. The cyclic order matters: the same four points listed in a different cyclic order may give determinants of mixed signs, and then the closed polygon they trace is not convex. The fact deferred there is supplied by Chapter 4 (Exercise 4.8(a),(b)): under strict convex position that union is the convex hull of the four points, and each q_k is a vertex of it.

The *shoelace formula* then computes its area as $\frac{1}{2}\left|\sum_k \det(q_k, q_{k+1})\right|$. Both triangles are positively oriented:

$$\det(q_1-q_0, q_2-q_0) = \det(q_1-q_0, q_2-q_1) = \delta_0 > 0,$$
$$\det(q_2-q_0, q_3-q_0) = \det(q_3-q_2, q_0-q_2) = \det(q_3-q_2, q_0-q_3) = \delta_2 > 0,$$

where the middle equality is invariance of the signed doubled area under a cyclic permutation of the three vertices and the others are instances of $\det(u,v) = \det(u, v+\alpha u)$. So their areas add without cancellation, and by bilinearity and alternation of det,

$$\det(q_1-q_0, q_2-q_0) + \det(q_2-q_0, q_3-q_0) = \sum_k \det(q_k, q_{k+1}),$$

the two terms $\det(q_2, q_0)$ and $\det(q_0, q_2)$ cancelling.

The following is Proposition 1.38, stated in Chapter 1 and deferred to this chapter for its construction.

Proposition 13.6 (Four-centre hull infimum)**.** *Over all quadruples of unit squares with pairwise disjoint interiors whose centres are in strict convex position in some cyclic order, the infimum of the area of the convex hull of the centres is* 0. *This holds even when restricted to axis-parallel unit squares that are pairwise disjoint as closed sets. The infimum is not attained.*

Proof. Fix $L > 1$ and $\varepsilon > 0$, and take axis-parallel unit squares centred at

$$p_0 = (0,0), \qquad p_1 = (L,\varepsilon), \qquad p_2 = (2L,-\varepsilon), \qquad p_3 = (3L,0).$$

Admissibility, in the strong form. The horizontal projections of the four squares are the intervals of length 1 centred at $0, L, 2L, 3L$. Any two of these centres differ by at least $L > 1$ in the first coordinate, so the corresponding intervals are disjoint; hence the closed squares are pairwise disjoint, and a fortiori their interiors are.

Strict convex position. In the cyclic order p_0, p_2, p_3, p_1 the four consecutive difference vectors are

$$p_2 - p_0 = (2L, -\varepsilon), \qquad p_3 - p_2 = (L, \varepsilon),$$
$$p_1 - p_3 = (-2L, \varepsilon), \qquad p_0 - p_1 = (-L, -\varepsilon),$$

and the four turn determinants are

$$\begin{aligned}
\det\big((2L, -\varepsilon), (L, \varepsilon)\big) &= 2L\varepsilon + L\varepsilon = 3L\varepsilon,\\
\det\big((L, \varepsilon), (-2L, \varepsilon)\big) &= L\varepsilon + 2L\varepsilon = 3L\varepsilon,\\
\det\big((-2L, \varepsilon), (-L, -\varepsilon)\big) &= 2L\varepsilon + L\varepsilon = 3L\varepsilon,\\
\det\big((-L, -\varepsilon), (2L, -\varepsilon)\big) &= L\varepsilon + 2L\varepsilon = 3L\varepsilon,
\end{aligned}$$

all strictly positive. So the four centres are in strict convex position with counterclockwise boundary order p_0, p_2, p_3, p_1.

The order matters. In the naive order p_0, p_1, p_2, p_3 the consecutive differences are (L, ε), $(L, -2\varepsilon)$, (L, ε), $(-3L, 0)$, and the four turn determinants are

$$-3L\varepsilon, \qquad +3L\varepsilon, \qquad +3L\varepsilon, \qquad -3L\varepsilon :$$

mixed signs, so that closed quadrilateral is not convex: it crosses itself. This is why strict convex position was defined relative to a cyclic order.

Area. Apply the shoelace formula in the boundary order p_0, p_2, p_3, p_1. The two terms involving $p_0 = (0, 0)$ vanish, and

$$\det(p_2, p_3) = \det\big((2L, -\varepsilon), (3L, 0)\big) = 3L\varepsilon,$$
$$\det(p_3, p_1) = \det\big((3L, 0), (L, \varepsilon)\big) = 3L\varepsilon,$$

so the hull area is $\frac{1}{2}|0 + 3L\varepsilon + 3L\varepsilon + 0| = 3L\varepsilon$, which tends to 0 as $\varepsilon \to 0$ with L fixed. Hence the infimum is 0.

Not attained. A quadruple in strict convex position has all four turn determinants strictly positive, so no three of the points are collinear and the hull is a convex quadrilateral of positive area. So the value 0 is not achieved. $\square$

Consistency with Theorem 1.1. Once $3L\varepsilon < \frac{1}{2}$, every one of the four sub-triangles of the quadruple has area less than the hull area, hence less than $\frac{1}{2}$; so by Theorem 1.1 every sub-triangle must be obtuse. The direct verification is the inner product at the middle vertex of p_0, p_1, p_2.

Four centres: the statement, not the method. The constructions of Chapters 5–12 — branches, ownership, the freeze, the minimisation, the multipliers — extend to four centres; what does not hold is the conclusion.

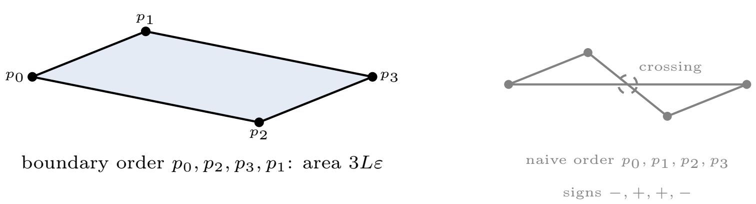


Figure 13.5: Proposition 13.6. The four centres in the correct boundary order form a thin convex quadrilateral of area $3L\varepsilon$; the inset shows the same four points joined in the naive order $p_0p_1p_2p_3$, which crosses itself. The vertical scale is exaggerated: the quadrilateral has height 2ε and width $3L > 3$.

These four facts delimit Theorem 1.1. With two centres, disjointness yields only the distance bound $\|P_2 - P_1\| \geq 1$ of (F1). With four centres nothing survives, by Proposition 13.6, even for axis-parallel unit squares that are disjoint as closed sets. For three centres, non-obtuseness separates the sharp constant $\frac{1}{2}$ of Proposition 1.36 from the degeneration of Example 1.37.

13.9 What a machine-checked proof establishes

Theorem 1.1 has been formally verified. This section states what the verification establishes. No formal syntax appears; the vocabulary below is the minimum needed to state the claims precisely.

What a proof assistant does. The verification is in Lean 4, over its mathematical library mathlib. In such a system a formal proof of a statement is a term whose *type* is that statement, and a small trusted kernel checks that the term has that type. The commands a user writes only build the term; they cannot cause an ill-typed term to be accepted. So the trusted base stays small even when the development is large: only the kernel must be believed, not the thousands of lines that produced the term.

What "no `sorry`" means. The marker `sorry` stands for an admitted gap: a place where the author asserts a statement without proving it. A development that compiles with no `sorry` anywhere contains no admitted gaps. It does not say that the statement proved is the statement intended.

What printed axioms mean. Asking the system to print the axioms of a theorem lists the assumptions its proof term ultimately depends on. Reporting only propositional extensionality, the axiom of choice in its classical form, and the soundness axiom for quotients means that nothing beyond the standard classical and quotient principles already used throughout mathlib was added: no auxiliary fact was postulated along the way.

Wrapper theorems. The risk in a formalisation is a subtly different, or vacuous, statement rather than a wrong proof. The mitigation is a thin *wrapper* theorem which restates the result in the paper's logical form — unit squares given by orthonormal side normals, disjointness of interiors, non-obtuseness as the three squared-side-length inequalities, area as the halved absolute shoelace determinant — with every project-level definition unfolded, and derives it from each entry point. The wrappers here are the following two theorems.

`PaperProofs.paper_center_area_lemma`
derived from `CenterAreaLemma.center_area_lemma`;

`PaperProofs.paper_center_area_lemma_reciprocal`
derived from `CenterAreaLemma.Reciprocal.center_area_lemma_reciprocal`.

The all-angle bound of Proposition 1.35 is formalised in the same library, as

`PaperProofs.paper_all_angle_bound`.

What is not established. Three limitations remain. That the formal definitions capture the informal notions is a human judgement; the wrappers mitigate it but do not eliminate it. That the kernel, the compiler and the hardware are correct is assumed. And formalisation says nothing about whether the prose proof in this book is correct: the prose proof is a separate document, organised to mirror the formal development module by module.

The module map is as follows.

module	**this book**
`FixedBranches`	owned separating branches (Ch. 5)
`Extremal`	the compact minimum (Ch. 9)
`CompleteContact`, `FirstVariation`	complete contact (Ch. 9)
`Farkas`, `Stationarity`	Farkas and the reciprocal equations (Chs. 8, 10)
`Forces` and its conclusions	the sine system and the nonpositive-sine case (Ch. 10)
`Cyclic`, `CyclicGeometry`	the cyclic case (Ch. 11)
`Transitive`, `TransitiveGeometry`	the transitive case (Ch. 12)
`OwnerConclusion`, `Proof`	the eight owner assignments and the assembly (Ch. 13)

The development compiles with no `sorry`, and the printed axioms of the wrapper theorems report only propositional extensionality, classical choice and quotient soundness.

What is not formalised. The equality construction (Proposition 1.36), the obtuse degeneration (Example 1.37) and the four-centre hull infimum (Proposition 13.6) are not formalised: each is an explicit construction requiring interior computations of concrete squares. So §§13.5, 13.6 and 13.8 of this chapter are human-checked only, whereas every step of the proof of Theorem 1.1 itself is covered.

13.10 The limits of pairwise information

Worked Example 13.B: a non-obtuse triple that is no centre triple

Pairwise centre separation is the only consequence of pairwise disjointness that constrains two centres at a time, and it is sharp: any two points at distance at least 1 are the centres of two unit squares with disjoint interiors. The following triple shows that it is not enough to prove Theorem 1.1. Take

$$B_0 = \left(\tfrac{1}{20},\ 1\right), \qquad C_0 = \left(-\tfrac{19}{20},\ \tfrac{4}{5}\right).$$

Verification, with exact fractions. Write $O = (0, 0)$.

- $\|B_0\|^2 = \frac{1}{400} + 1 = \frac{401}{400} > 1$.
- $\|C_0\|^2 = \frac{361}{400} + \frac{256}{400} = \frac{617}{400} > 1$.
- $B_0 - C_0 = (1, \frac{1}{5})$, so $\|B_0 - C_0\|^2 = 1 + \frac{1}{25} = \frac{26}{25} > 1$.
- Non-obtuse at O: $\langle B_0, C_0\rangle = -\frac{19}{400} + \frac{320}{400} = \frac{301}{400} > 0$.
- Non-obtuse at B_0: $\langle O - B_0,\ C_0 - B_0\rangle = \big\langle(-\frac{1}{20}, -1),\ (-1, -\frac{1}{5})\big\rangle = \frac{1}{20} + \frac{1}{5} = \frac{1}{4} > 0$.
- Non-obtuse at C_0: $\langle O - C_0,\ B_0 - C_0\rangle = \big\langle(\frac{19}{20}, -\frac{4}{5}),\ (1, \frac{1}{5})\big\rangle = \frac{95}{100} - \frac{16}{100} = \frac{79}{100} > 0$.
- $\det(B_0, C_0) = \frac{1}{20} \cdot \frac{4}{5} - 1 \cdot \left(-\frac{19}{20}\right) = \frac{4}{100} + \frac{95}{100} = \frac{99}{100}$, so the area is $\frac{99}{200} < \frac{1}{2}$.

So (O, B_0, C_0) is a non-obtuse triangle with all sides at least 1 and area $\frac{99}{200} < \frac{1}{2}$. By Theorem 1.1 it is therefore not the centre triple of three admissible unit squares, even though all three pairwise distances are at least 1. No argument using only pairwise distances can prove Theorem 1.1; Exercise 13.5 quantifies the gap.

13.11 Retrospect: which hypothesis did which work

Disjoint interiors. Used only through (F1), (F3) and (F4): each pair certifies its own disjointness by a single owned branch, of threshold $H \geq 1$, and centre separation (F1) supplies the distinctness of the three centres that lets one speak of angles. That last use is dispensable, since Cauchy–Schwarz against the branch already gives $\|d_i\| \geq \langle d_i, n_i\rangle = H_i \geq 1$, but it is what is invoked in (C1) and in the all-angle bound. After the freeze, disjointness never appears again: everything from Chapter 5 onwards is optimisation applied to three linear inequalities.

Unit side length. Sets the scale twice: it makes every side of the centre triangle at least 1, by the Cauchy–Schwarz estimate just quoted, and it makes every threshold at least 1. The second is the quantitative content of the proof; it rests on one scalar fact from Chapter 2: for a unit vector with coordinates a, b in an orthonormal basis, $|a| + |b| \geq 1$.

Non-obtuseness. Used in three places, all before the final step:

(i) noncollinearity, so that $D > 0$ and σ is defined;
(ii) the subcritical triangle lemma (F5), which bounds the sides by $\sqrt{2}$ and hence makes the feasible set bounded, and which makes the minimiser strictly acute;
(iii) acuteness as an open condition, used for persistence under small motions in the proof of complete contact and in checking the second hypothesis of the Farkas lemma.

Consequently non-obtuseness plays no role in Chapters 10–12: once complete contact holds, the case analysis uses only $H_i \geq 1$, $\lambda_i \geq 0$, the sine system and ownership.

Ownership. Carries no metric information, the threshold being owner-independent by (F4), but it carries discrete information, and is used for nothing else. It defines the tournament, which organises the case analysis; and in the transitive case it forces two of the normals to be axes of a single frame, hence parallel or perpendicular. Positivity of the sines excludes parallel, so they are perpendicular and $s_3 = 1$, collapsing three parameters to one.

The number 2. The target $\sum_i \lambda_i H_i \geq 2$ comes from Euler's identity in degree 2: the objective is a quadratic form in x, so $\langle g, x\rangle = 2D$, while each constraint is linear, so $\langle a_i, x\rangle = b_i(x)$, and complete contact (C5) makes that H_i. Worked Example 13.A exhibits the 2 realised as $1 \cdot 1 + 0 \cdot 1 + 1 \cdot 1$: two contacts, each with unit threshold and unit multiplier, and one contact contributing nothing. Dividing by the factor 2 of Euler's identity and the factor 2 of doubled area gives the constant $\frac{1}{2}$ of Theorem 1.1.

The four layers. Geometry (Ch. 5) replaces each pair's disjointness by one owned branch. Optimisation (Chs. 6–9) produces a minimiser at which every branch is active. Duality (Chs. 8, 10) turns activity into nonnegative multipliers and a reciprocal force diagram. Combinatorics (Chs. 10–12) reduces the force geometry to two scalar inequalities.

What the proof does not give. No classification of extremal configurations; no uniqueness of the minimiser; no equality analysis inside the optimisation, so no answer to the question of when $\sum_i \lambda_i H_i = 2$; and, the argument being by contradiction, no direct information about any particular admissible triple. That the constant $\frac{1}{2}$ is sharp is known only from Proposition 1.36, an explicit construction, and not from the proof.

Exercises

Exercise 13.1 (Bringing an owner assignment to normal form)**.** Let Δ be a contact datum with owner assignment $(o_1, o_2, o_3) = (2, 2, 1)$.

(a) Draw $\mathcal{T}(\Delta)$ and identify its source, middle vertex and sink.

(b) Find the $\pi \in S_3$ for which $\mathcal{T}(\Delta^\pi)$ is representative (b) of Lemma 13.4. (Note the inverse: the equation to solve is $\pi^{-1} \cdot \mathcal{T}(\Delta) =$ representative (b).)

(c) Write this π as a product of the generators ρ and $\tau = (2\ 3)$, and use Lemma 13.2 twice to compute σ', n_1', n_2', n_3', H_i', λ_i' and s_1', s_2', s_3' in terms of the original data. Check directly that $N(\Delta^\pi) = N(\Delta)$.

(d) Is $\sigma' = \sigma$? Explain briefly why the answer does not affect the argument.

Exercise 13.2 (Left and right)**.** Verify that $\pi \mapsto \Delta^\pi$ is a right action of S_3, that is, that $\Delta^{\pi\rho} = (\Delta^\pi)^\rho$ and not in general $(\Delta^\rho)^\pi$; and that $\alpha \mapsto \alpha \cdot \mathcal{T}$ is a left action. Then show that for any group G acting on a set on the right, $g \cdot x := x^{g^{-1}}$ defines a left action, and that this is exactly the passage that produces the π^{-1} in Lemma 13.4. Finally, exhibit $\pi, \rho \in S_3$ and a contact datum for which $(\Delta^\pi)^\rho \neq (\Delta^\rho)^\pi$, so that the distinction is not vacuous.

Exercise 13.3 (A contact datum with no problem behind it)**.** Definition 13.1 makes no reference to a minimisation problem. Take $\sigma = +1$ and let n_1, n_2, n_3 be the unit vectors at angles 0, $2\pi/3$, $4\pi/3$, with n_i an axis of the frame F_i and $o_i = i$; this is the configuration of Example 10.19.

(a) Compute H_1, H_2, H_3 and s_1, s_2, s_3, and solve the sine system for $\lambda_1, \lambda_2, \lambda_3$.

(b) Recover the centres from the reciprocal equations (put $P_1 = 0$) and check that every clause of Definition 13.1 holds, so that the result is a contact datum.

(c) Compute $D(\Delta)$ and $N(\Delta)$ and verify $2D = N \geq 2$ directly.

(d) Show that this datum arises from no assumed counterexample: its doubled area exceeds 1, so it is not a feasible point of any relaxed problem of the kind built in §13.1. Explain why Proposition 13.5 nevertheless applies to it.

Exercise 13.4 (A redundant clause)**.** Clause (a) of Definition 13.1 demands $D(\Delta) > 0$. Show that this demand is redundant, as follows. Let Δ satisfy (b), (c) and (d), together with the requirement in (a) that $\sigma \in \{-1, +1\}$ and that $d_1 + d_2 + d_3 = 0$, but not necessarily $D(\Delta) > 0$.

(a) Derive the sine system and the identity $2D(\Delta) = N(\Delta)$, checking that neither derivation uses the sign of D.

(b) Go through §§13.2.2–13.2.4 and verify that positivity of D is only ever transported from Δ to Δ^π, never used to prove anything.

(c) Conclude that $N(\Delta) \geq 2$ and hence $D(\Delta) \geq 1 > 0$.

Why, then, is the clause there? (What would go wrong in observation (i) of §13.2.1 if it were dropped?)

Exercise 13.5. Show that among all non-obtuse triangles with all sides of length at least 1, the minimum area is $\sqrt{3}/4$, attained only by the equilateral triangle of side 1. (Hint: let θ be the largest angle; then $\theta \in [\pi/3, \pi/2]$.) Deduce that pairwise separation of the centres alone — the only consequence of disjointness that constrains two centres at a time — cannot yield a bound better than $\sqrt{3}/4 < \frac{1}{2}$, so that Theorem 1.1 is stronger than anything obtainable pairwise. Where does the triple of Worked Example 13.B sit relative to the two bounds $\sqrt{3}/4$ and $\frac{1}{2}$?

Exercise 13.6. Show that a triangle contained in a closed $w \times h$ rectangle has area at most $wh/2$. (Hint: split it by the vertical line through the vertex with middle x-coordinate.) Now suppose three unit squares with pairwise disjoint interiors have all three centres inside a $w \times h$ rectangle, and that their centre triangle is non-obtuse. Deduce from Theorem 1.1 that $wh \geq 1$. This is the *wide branch* of the three-centre packing threshold mentioned in Chapter 1; the full threshold, and the equivalence of its wide branch with Theorem 1.1, are beyond this book.