Counting unit squares by their centers: sharp convex bounds and exact strip laws

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Abstract

$N_{\max}$ counts interior-disjoint unit squares, in any orientation, with all centers in a given region. For compact convex K, at most $\operatorname{area}(K) + w_x(K) + w_y(K) + 1 = \operatorname{area}(K \oplus Q_0)$ centers fit, Q_0 the centered axis-parallel unit square: the trivial axis-parallel bound survives arbitrary rotations. Specializing, $N_{\max}(a, b) \le (a + 1)(b + 1)$, with equality for integer a, b. The rectangle case uses a weighted boundary score and is fully formalized in Lean 4 over mathlib; the convex case uses a cubical clipping lemma paying in ℓ^1 boundary length and exterior turning. We prove the exact strip law $\lfloor\sqrt{w^2 + h^2}\rfloor + 1$ for $w \le (\sqrt{2} - 1)/2$ (also formalized), extended to $w \le \xi$ when $w^2 + h^2 < 4$; the three-branch threshold $T_3(w)$; and, for $1/\sqrt{2} \le w < 1$, exact bands with thresholds $\tau_{2r+1} = r/w$, $\tau_{2r+2} = r/w + \sqrt{1 - w^2}$. Sharp local bounds include the exact near-wall gap, the corner threshold $1 + 1/(2\sqrt{2})$, exactness of $g(d) = \sqrt{1 - d^2}$, and the failure of three natural strengthenings. This paper was produced with a mix of ChatGPT and Claude Fable and has not yet been adequately human reviewed.

1 Introduction

How many interior-disjoint unit squares can have all their centers in a prescribed region? A unit square is closed, of side 1, in any orientation; a family is interior-disjoint if no two interiors meet, boundary contact allowed; only the centers are constrained, and the squares may protrude from the region. For $a, b \ge 0$ let $N_{\max}(a, b)$ be the supremum of $|F|$ over finite interior-disjoint families F of unit squares with all centers in $[0, a] \times [0, b]$; by Theorem 1.2 below every admissible family has at most $(a + 1)(b + 1)$ members, so the supremum is an attained maximum. Center-count bounds of this kind serve as pruning rules in computer searches for packing impossibility results.

The companion function $S_{\max}(a, b)$ is the maximum number of interior-disjoint unit squares *contained* in $[0, a] \times [0, b]$, and $s(n) = \inf\{t : S_{\max}(t, t) \ge n\}$ is the side of the smallest square containing n interior-disjoint unit squares; the study of $s(n)$ goes back to Erdős and Graham [1], with known exact values and dense packings surveyed and established in [2, 5, 12, 9, 3]. The two functions are linked, for $a, b \ge 1$, by

$$S_{\max}(a, b) \le N_{\max}(a - 1, b - 1), \qquad N_{\max}(a, b) \le S_{\max}(a + \sqrt{2},\, b + \sqrt{2}), \tag{1}$$

and trivially $N_{\max}(a, b) \ge S_{\max}(a, b)$: the center of a contained square lies at distance $\ge 1/2$ from each side, and a center at distance $\ge 1/\sqrt{2}$ from each side leaves room for the square in every orientation. These inequalities are the interface to the $s(n)$ literature and are not used further.

Our first result bounds the count for an arbitrary convex center region. Write $Q_0 = [-1/2, 1/2]^2$ for the axis-parallel unit square centered at the origin, $D(x, \rho)$ for the closed disk of radius ρ about x, $w_x(K)$ and $w_y(K)$ for the coordinate widths of a convex set K, and $K \oplus Q_0 = \{k + q : k \in K,\ q \in Q_0\}$ for the Minkowski sum.

Theorem 1.1 (Convex center count)**.** *Let $K \subset \mathbb{R}^2$ be compact and convex. If N interior-disjoint unit squares have all their centers in K, then*

$$N \le \operatorname{area}(K \oplus Q_0) = \operatorname{area}(K) + w_x(K) + w_y(K) + 1.$$

If all squares were axis-parallel, each would be $c_i + Q_0$ with $c_i \in K$, the squares would be interior-disjoint subsets of $K \oplus Q_0$, and area comparison would give the bound at once. A tilted square centered in K is only contained in $K \oplus D(0, \sqrt{2}/2)$, and $\operatorname{area}(K \oplus D(0, \sqrt{2}/2)) > \operatorname{area}(K \oplus Q_0)$, the excess being $\pi/2 - 1$ when K is a point. The content of Theorem 1.1 is that the axis-parallel bound survives arbitrary independent rotations. For point sets with pairwise distances ≥ 1 in convex K, Groemer and Oler proved $N \leq (2/\sqrt{3}) \operatorname{area}(K) + \frac{1}{2} \operatorname{per}(K) + 1$ [4, 10]; our area coefficient for squares is 1.

Theorem 1.2 (Rectangle center bound). *For all $a, b \geq 0$, $N_{\max}(a, b) \leq (a + 1)(b + 1)$, with equality when $a, b \in \mathbb{N}$, attained by axis-parallel squares centered at the integer grid points.*

The axis-parallel case is folklore-adjacent: ℓ^∞ separation of centers plus counting the half-open unit grid cells meeting the rectangle gives at most $(\lfloor a \rfloor + 1)(\lfloor b \rfloor + 1)$.

For narrow strips we prove exact laws. Set $\tau = (\sqrt{2}-1)/2$ and $\xi = (4\sqrt{2}-2\sqrt{3})/5$, the smaller positive root of $5w^2 - 8\sqrt{2}\, w + 4 = 0$, equivalently of $\sqrt{4 - w^2} + 2w = 2\sqrt{2}$, with $\xi < 1/\sqrt{2}$.

Theorem 1.3 (Very-narrow strip law). *If $0 \leq w \leq \tau$ and $h \geq 0$, then $N_{\max}(w, h) = \left\lfloor \sqrt{w^2 + h^2} \right\rfloor + 1$.*

Corollary 1.4 (First diagonal threshold). *If $0 \leq w \leq \xi$ and $D = \sqrt{w^2 + h^2} < 2$, then $N_{\max}(w, h) = \lfloor D \rfloor + 1$.*

We say "n centers fit in $[0, w] \times [0, h]$" to abbreviate $N_{\max}(w, h) \geq n$. Let $T_3(w) = \inf\{h \geq 0 : N_{\max}(w, h) \geq 3\}$ for $0 \leq w < 1$.

Theorem 1.5 (Three-center threshold). *For $0 \leq w < 1$,*

$$T_3(w) = \begin{cases} \sqrt{4 - w^2}, & 0 \leq w \leq \xi, \\ 2\sqrt{2} - 2w, & \xi \leq w \leq 1/\sqrt{2}, \\ 1/w, & 1/\sqrt{2} \leq w < 1, \end{cases}$$

and the infimum is attained in every branch: three centers fit in $[0, w] \times [0, h]$ if and only if $h \geq T_3(w)$.

Theorem 1.6 (Exact wide bands). *Let $1/\sqrt{2} \leq w < 1$ and $d = \sqrt{1 - w^2}$, and define $\tau_1 = 0$, $\tau_{2r+1} = r/w$ for $r \geq 1$, and $\tau_{2r+2} = r/w + d$ for $r \geq 0$. Then for every $n \geq 1$,*

$$\tau_n \leq h < \tau_{n+1} \quad \Longrightarrow \quad N_{\max}(w, h) = n.$$

The center-area lemma of [7] (three interior-disjoint unit squares with non-obtuse center triangle have center-triangle area $\geq 1/2$) is used as an input to Theorems 1.5 and 1.6 and is not reproved here.

Integer rectangles attain the bound of Theorem 1.2. The very-narrow law fails for every $w > \xi$ at h just below $\sqrt{4 - w^2}$ (details in Section 5). The toolbox of Section 7 collects sharp local lemmas, several of which show that natural strengthenings of our tools fail.

Two of the main results, Theorem 1.2 with its sharpness and Theorem 1.3, are machine-checked end to end in Lean 4, and every closed-form constant and inequality in the paper was verified by independent numerical audit; see Section 9.

The proofs run as follows. Theorem 1.2 is proved in Section 3 by charging each square a score, area inside the rectangle plus weighted boundary trace and covered corners, and showing each square scores at least 1. Section 4 proves Theorem 1.1 via a cubical clipping lemma: whatever the region fails to contain of a square it pays for in ℓ^1 boundary length and exterior turning inside the square. Section 5 proves Theorem 1.3 by packing ordered chords along the strip, then develops rounded neighbourhoods and three positional obstructions that give the narrow branches of Theorem 1.5 and Corollary 1.4. Section 6 handles the wide branch via the center-area input, completes Theorem 1.5, and proves Theorem 1.6 with sheared constructions matching the gap bookkeeping. Section 7 presents the toolbox, Section 8 the open problems, and Section 9 the Lean validation.

2 Preliminaries

This section fixes terminology and records four elementary facts used throughout: the support-radius formula, the center-distance lemma, its sharp vertical-gap corollary, and a disjointness criterion for squares with a common frame.

A *unit square* is a closed square of side 1 in the plane, in any orientation. A family of unit squares is *interior-disjoint* if no two of its members have intersecting interiors; boundary contact is allowed. We write $c(S)$ for the center of a square S. Every unit square S contains its incircle $D(c(S), \frac{1}{2})$ and is contained in its circumcircle $D(c(S), \frac{\sqrt{2}}{2})$. More precisely, for a unit vector e making angle θ with a side normal of S, the support radius $\max_{p\in S}(p - c(S)) \cdot e$ equals

$$q(\theta) = \tfrac{1}{2}\big(|\cos\theta| + |\sin\theta|\big) \in \Big[\tfrac{1}{2}, \tfrac{\sqrt{2}}{2}\Big]. \tag{2}$$

Lemma 2.1 (Center distance)**.** *Two interior-disjoint unit squares have centers at distance at least* 1*.*

Proof. If the centers were at distance less than 1, the two open incircles of radius $\frac{1}{2}$ would meet, hence so would the square interiors. □

Corollary 2.2 (Vertical gap)**.** *Let S_1, S_2 be interior-disjoint unit squares with centers (x_1, y_1) and (x_2, y_2). If $|x_1 - x_2| \le w \le 1$, then $|y_1 - y_2| \ge \sqrt{1 - w^2}$. The bound is sharp for every $w \in [0, 1]$: two unit squares sharing a full edge, with common side direction $(w, \sqrt{1 - w^2})$, attain it.*

Proof. By Lemma 2.1, $(y_1 - y_2)^2 \ge 1 - (x_1 - x_2)^2 \ge 1 - w^2$. For sharpness, set $v = (w, \sqrt{1 - w^2})$, let u be a unit vector perpendicular to v, and take

$$S_1 = \big\{sv + tu : |s|, |t| \le \tfrac{1}{2}\big\}, \qquad S_2 = S_1 + v.$$

The centers 0 and v satisfy $|x_1 - x_2| = w$ and $|y_1 - y_2| = \sqrt{1 - w^2}$. The squares share the edge $\big\{\frac{1}{2}v + tu : |t| \le \frac{1}{2}\big\}$, and their interiors lie in the two opposite open half-planes bounded by that edge's line, so they are interior-disjoint. □

Several constructions below assemble squares sharing a single frame; the following criterion certifies their disjointness once and for all.

Lemma 2.3 (Common-axis criterion)**.** *Let S_1, S_2 be unit squares with common side axes: $S_i = c_i + \{su + tv : |s|, |t| \le \frac{1}{2}\}$ for a fixed orthonormal pair u, v. If $|(c_1 - c_2) \cdot u| \ge 1$ or $|(c_1 - c_2) \cdot v| \ge 1$, then the interiors of S_1 and S_2 are disjoint.*

Proof. Suppose $|(c_1 - c_2) \cdot u| \ge 1$. The orthogonal projection of $\operatorname{int} S_i$ onto the u-axis is the open interval of length 1 centered at $c_i \cdot u$; these two intervals are disjoint, hence so are the interiors. □

Two properties of $N_{\max}$ are immediate from the definition and are used without further comment: $N_{\max}(a, b) = N_{\max}(b, a)$, by reflection in the diagonal, and $N_{\max}$ is nondecreasing in each argument, since enlarging the rectangle admits every earlier configuration of centers.

We record the one-center case: $N_{\max}(w, h) \ge 1$ always, and $N_{\max}(w, h) = 1$ exactly when $\sqrt{w^2 + h^2} < 1$. Indeed, when the diagonal is shorter than 1, Lemma 2.1 forbids a second center, while when $\sqrt{w^2 + h^2} \ge 1$, the shared-edge pair of Corollary 2.2 with side direction $(w, h)/\sqrt{w^2 + h^2}$ places two centers at distance 1 inside $[0, w] \times [0, h]$.

3 The rectangle bound

We prove Theorem 1.2 by charging each square of a packing a score against the rectangle and summing the charges. The proof does not replace tilted squares by axis-parallel ones: in general no replacement preserves both the centers and interior-disjointness. For related work on packing unit squares in a rectangle see Nagamochi [9].

Definition 3.1 (Rectangle score)**.** For a unit square Q and a closed axis-parallel rectangle R set

$$\sigma_R(Q) \;=\; \mathrm{area}(Q\cap R) \;+\; \tfrac{1}{2}\,\mathrm{length}(Q\cap\partial R) \;+\; \tfrac{1}{4}\,\#\{v:\ v \text{ a vertex of } R,\ v\in\mathrm{int}\,Q\}, \tag{3}$$

where the middle term is one-dimensional length along the sides of R.

Interior area counts in full, the boundary trace at weight $\frac{1}{2}$, and each covered corner at weight $\frac{1}{4}$: the boundary terms compensate the part of Q clipped away by R.

Lemma 3.2 (One-square score)**.** *If $c(Q)\in R$, then $\sigma_R(Q)\geq 1$.*

Proof. The plan: reduce to general position by an outward perturbation, identify σ_R with a product measure, factor that measure over the four quadrants at the center, and show that each quadrant contributes at least $\frac{1}{4}$. Translate so that $c(Q)$ is the origin and write $R=[-\alpha,\beta]\times[-\gamma,\delta]$ with $\alpha,\beta,\gamma,\delta\geq 0$.

Step A: reduction to general position. Call the configuration *generic* if no side line of R contains a side of Q, no vertex of R lies on ∂Q, and $\alpha,\beta,\gamma,\delta>0$. Suppose the lemma is proved for generic configurations. In the general case, move each side of R outward by a small generic amount: the perturbed rectangle $R_\varepsilon\supseteq R$ is generic. The outward direction matters twice: it keeps the center inside R_ε, and it makes any vanishing side parameters positive, so the quadrant decomposition below is nondegenerate. Moreover

$$\limsup_{\varepsilon\to 0}\sigma_{R_\varepsilon}(Q)\;\leq\;\sigma_R(Q),$$

since the area term converges, and any losses in the limit are boundary or corner terms already present in $\sigma_R(Q)$. Hence $\sigma_R(Q)\geq\limsup_{\varepsilon\to 0}\sigma_{R_\varepsilon}(Q)\geq 1$, and we may assume the configuration is generic.

Step B: product-measure identity. For $u<v$ let

$$\mu_{[u,v]}\;=\;\lambda_{[u,v]}+\tfrac{1}{2}\delta_u+\tfrac{1}{2}\delta_v,$$

Lebesgue measure on $[u,v]$ plus half-atoms at the endpoints, δ_t denoting the Dirac mass at t. In a generic configuration

$$\sigma_R(Q)\;=\;\bigl(\mu_{[-\alpha,\beta]}\otimes\mu_{[-\gamma,\delta]}\bigr)(Q).$$

Indeed, $\lambda\otimes\lambda$ gives $\mathrm{area}(Q\cap R)$; each product of an endpoint atom with λ gives half the length of the trace of Q on the corresponding side of R; and each product of two atoms gives $\frac{1}{4}$ at a vertex of R, which by genericity lies in Q exactly when it lies in $\mathrm{int}\,Q$.

Step C: quadrant split. Since $\alpha,\beta,\gamma,\delta>0$, neither factor has an atom at 0, so the coordinate axes are null for the product measure and the four closed quadrants at the origin contribute additively. It therefore suffices to show that each quadrant's part of Q receives mass at least $\frac{1}{4}$.

Step D: one quadrant. The restriction of the product measure to the closed positive quadrant is $\mu^+_{[0,\beta]}\otimes\mu^+_{[0,\delta]}$, where $\mu^+_{[0,t]}=\lambda_{[0,t]}+\frac{1}{2}\delta_t$ retains only the atom at the outer endpoint. Reflections in the coordinate axes carry the other three quadrants to this one; they replace (β,δ) by a pair (u,v) with $u\in\{\alpha,\beta\}$, $v\in\{\gamma,\delta\}$, and Q by a reflected copy, still a unit square centered at the origin. By the symmetries of that square we may take its angle $\theta\in[0,\pi/4]$; write $c=\cos\theta$, $s=\sin\theta$. The part of Q in the closed positive quadrant is then the quadrilateral P with vertices

$$O=(0,0),\qquad A=\Bigl(\frac{1}{2c},0\Bigr),\qquad B=\Bigl(\frac{c-s}{2},\frac{c+s}{2}\Bigr),\qquad C=\Bigl(0,\frac{1}{2c}\Bigr),$$

of area exactly $\frac{1}{4}$ (a shoelace computation). The quadrant contribution is the score of P clipped by the window $[0,u]\times[0,v]$: the area of $P\cap([0,u]\times[0,v])$, plus half the length of the trace of P on each of the lines $x=u$ and $y=v$ inside the window, plus $\frac{1}{4}$ if $(u,v)\in P$. We show this is at least $\frac{1}{4}$ for all $u,v\geq 0$; three cases cover all configurations.

Case 1: $(u,v)\in P$. The corner atom alone contributes $\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$.

Case 2: $u\leq(c-s)/2$ and (u,v) above P, that is, $v>(\frac{1}{2}+su)/c$. For $x\leq(c-s)/2$ the quadrilateral P is bounded above by the side CB, on the line $-xs+yc=\frac{1}{2}$, so P meets the column $\{0\leq x\leq u\}$ in $\{0\leq y\leq(\frac{1}{2}+sx)/c\}$; the cut at $y=v$ removes nothing, the trace on $y=v$ is empty, and the corner atom does not act. The score equals

$$\int_0^u \frac{\frac{1}{2}+sx}{c}\,dx+\frac{1}{2}\cdot\frac{\frac{1}{2}+su}{c} \;=\; \frac{u}{2c}+\frac{su^2}{2c}+\frac{\frac{1}{2}+su}{2c},$$

the last summand being the half-weighted trace on the line $x=u$, of length $(\frac{1}{2}+su)/c$. This expression is increasing in u and at $u=0$ equals $1/(4c)\geq\frac{1}{4}$.

Case 3: the remaining configurations, $u>(c-s)/2$ and $(u,v)\notin P$. Here the parts of P outside the window are triangles cut off by the clipping lines, each covered by half of the slice that cuts it. The part of P to the right of $x=u$, present when u lies between the abscissae $(c-s)/2$ of B and $1/(2c)$ of A, is a triangle whose base is the vertical slice of P at $x=u$ and whose horizontal extent is at most $1/(2c)\leq\sqrt{2}/2<1$; its area is therefore at most half the length of its base, which is exactly the credited edge term at $x=u$. The part of P above $y=v$ is treated the same way, with vertical extent at most $(c+s)/2<1$. When both cuts act, the doubly cut corner region is covered twice, which only helps. The edge credits thus repay the clipped area, and the score is at least $\operatorname{area}(P)=\frac{1}{4}$.

Summing the four quadrant contributions gives $\sigma_R(Q)\geq 1$. The case analysis just completed is elementary but fiddly; it is carried out in full, with no appeal to pictures, in the Lean formalization (Section 9). □

Proof of Theorem 1.2. Suppose first that $a=0$ or $b=0$. All centers lie on a segment of length $m=\max(a,b)$, and by Lemma 2.1 they are pairwise at distance at least 1, so there are at most $\lfloor m\rfloor+1\leq(a+1)(b+1)$ of them.

Now let $a,b>0$, put $R=[0,a]\times[0,b]$, and let $Q_1,\ldots,Q_N$ be interior-disjoint unit squares with centers in R. Lemma 3.2 gives $N\leq\sum_{i=1}^N\sigma_R(Q_i)$, and we sum the three kinds of terms separately. The interiors of the Q_i are disjoint, so the area terms total at most $\operatorname{area}(R)=ab$. On each side of R the traces $Q_i\cap\partial R$ have pairwise disjoint relative interiors, since away from its endpoints each trace lies in the open square $\operatorname{int}Q_i$; the edge terms therefore total at most $\frac{1}{2}\cdot 2(a+b)=a+b$. Each vertex of R is interior to at most one Q_i, so the corner terms total at most $\frac{1}{4}\cdot 4=1$. Hence

$$N\;\leq\;ab+(a+b)+1\;=\;(a+1)(b+1).$$

For $a,b\in\mathbb{N}$ the axis-parallel unit squares centered at the $(a+1)(b+1)$ integer points of $[0,a]\times[0,b]$ are interior-disjoint, so $N_{\max}(a,b)=(a+1)(b+1)$. □

Remark 3.3. Theorem 1.2, including Lemma 3.2 and the integer-grid sharpness construction, is fully formalized in Lean 4, as is the very-narrow strip law. Section 9 states exactly what is and is not machine-checked.

4 The convex bound via cubical clipping

The cubical clipping lemma bounds the area a convex region fails to capture from a unit square centered in it, charging the loss to boundary length and turning inside the square. Subsection 4.1 proves the lemma; Subsection 4.2 sums it over a packing to prove Theorem 1.1.

4.1 The clipping lemma

Two boundary functionals are needed. For a rectifiable curve Γ let

$$P_1(\Gamma) = \int_\Gamma \big(|dx| + |dy|\big)$$

be its ℓ^1 length; for a convex body M the set function $A \mapsto P_1(\partial M \cap A)$ is a Borel measure on ∂M. For compact convex $C \subset \mathbb{R}^2$ with nonempty interior, let κ_C be the exterior-turning measure on ∂C, normalized so that $\kappa_C(\partial C) = 2\pi$; when C is a convex polygon, κ_C places at each vertex an atom equal to the exterior angle there.

Lemma 4.1 (Cubical clipping)**.** *Let S be a unit square with center o, in any orientation, and let $C \subset \mathbb{R}^2$ be compact convex with $o \in C$ and $\operatorname{int} C \neq \emptyset$. Then*

$$\operatorname{area}(S \setminus C) \;\le\; \tfrac{1}{2}\, P_1(\partial C \cap \operatorname{int} S) + \frac{\kappa_C(\operatorname{int} S)}{2\pi}. \tag{4}$$

Equivalently, since $\operatorname{area}(S) = 1$,

$$\operatorname{area}(S \cap C) + \tfrac{1}{2}\, P_1(\partial C \cap \operatorname{int} S) + \frac{\kappa_C(\operatorname{int} S)}{2\pi} \;\ge\; 1.$$

What the region fails to contain of the square, it must pay for in boundary length and turning inside the square.

Proof. The plan: prove (4) for polygons transverse to S, classifying the components of $S \setminus C$ as chord-separated (charged to boundary length through a cap–chord table) or wrapping (at most one, charged through a wedge inequality), then pass to general C by a dilation centered at o and outer polygonal approximation.

Step 1: transverse polygons and the component–arc correspondence. Assume first that C is a convex polygon transverse to S: no vertex of C lies on ∂S, no edge of C meets ∂S tangentially or along a segment, and $\partial C \cap \partial S$ is finite.

If $\partial C \subset \operatorname{int} S$, then $\kappa_C(\operatorname{int} S) = \kappa_C(\partial C) = 2\pi$, so the right side of (4) is at least $1 \ge \operatorname{area}(S \setminus C)$ and we are done. Otherwise, call a maximal subarc of ∂C with relative interior in $\operatorname{int} S$ and endpoints on ∂S a *crossing arc*; by transversality there are finitely many. By convexity of $C \cap S$, the crossing arcs are in bijection with the components of $S \setminus C$: each component U has as free boundary exactly one crossing arc Γ, with endpoints $p, q \in \partial S$ and relative interior in $\operatorname{int} S$. Nothing below depends on this bijection; a component with several free arcs would be charged through any one of them.

Step 2: each component is chord-separated or wrapping. Fix a component U with arc Γ and endpoints p, q. Since $p, q \in C$ and C is convex, $[p, q] \subset C$. Transversality excludes a side of S lying on the line pq; as $p, q \in \partial S$, the line pq therefore meets S exactly in $[p, q]$: the segment $[p, q]$ is the *full* chord of S on that line. The connected set U is disjoint from $C \supseteq [p, q]$, hence avoids this full chord and lies strictly on one side of the line pq. Let B be the chordal cap of C cut off by $[p, q]$ on the side of Γ: the part of C bounded by Γ and the chord. Then B is convex and U lies on the B-side of the line. Two cases.

Case (a): o lies weakly on the far side of the line pq from U. Then U is contained in the straight cap of S cut off by the line pq on the side of U, at signed distance $t \ge 0$ from o. Work in S-aligned coordinates, with the unit normal of the cutting line at angle $\theta \in [0, \pi/4]$ from a side normal of S; set $c = \cos\theta$, $s = \sin\theta$, $a = (c - s)/2$, $h = (c + s)/2$. The cap area A and the full chord length ℓ are:

	A	ℓ
$0 \le t \le a$	$\dfrac{1}{2} - \dfrac{t}{c}$	$\dfrac{1}{c}$
$a \le t \le h$	$\dfrac{(h-t)^2}{2sc}$	$\dfrac{h-t}{sc}$

The second row is void when $s = 0$, and the cap is empty for $t > h$. In both rows $A \le \ell/2$: in the first row this reads $c - 2t \le 1$, true since $c \le 1$; in the second, $h - t \le 1$, true since $h \le \sqrt{2}/2$. The chord endpoints are p and q, so $\ell = |p - q|_2$ and

$$\operatorname{area}(U) \;\le\; \tfrac{1}{2}\,|p - q|_2 \;\le\; \tfrac{1}{2}\,|p - q|_1 \;\le\; \tfrac{1}{2}\,P_1(\Gamma), \tag{5}$$

the last step because the total variation of each coordinate along Γ is at least the coordinate gap between its endpoints. Call a component charged by (5) *filled.*

Case (b): o lies strictly on the same side of the line pq as U. That side is the B-side, and B is the intersection of C with the closed half-plane on that side, so $o \in B$; by convexity $\operatorname{conv}\{p, o, q\} \subseteq B$. Call U *wrapping.* At most one component wraps. Suppose U_1, U_2 both wrap, with arcs Γ_1, Γ_2, chords $[p_i, q_i]$, and caps B_i. The arcs are disjoint subarcs of the closed convex curve ∂C, so they occur in cyclic order and Γ_2 lies in the complementary arc of ∂C determined by Γ_1; in particular the two chords do not cross. Let H_1 be the closed half-plane bounded by the line p_1q_1 that contains B_1. The complementary arc lies in the opposite closed half-plane, hence so do p_2, q_2, the chord $[p_2, q_2]$, and therefore the convex cap B_2, whose boundary is $\Gamma_2 \cup [p_2, q_2]$. But wrapping of U_1 places o in the interior of H_1, while wrapping of U_2 places $o \in B_2$, which misses the interior of H_1: a contradiction.

Step 3: the wrapping sector and its arc estimates. Suppose the component U wraps, with arc Γ and endpoints p, q. Write $e(\varphi) = (\cos\varphi, \sin\varphi)$, let $r(\varphi) = \max\{\rho \ge 0 : o + \rho e(\varphi) \in S\}$ be the radial function of S at o, and set $m(\varphi) = |\cos\varphi| + |\sin\varphi|$, so that $|x - o|_1 = r(\varphi)\, m(\varphi)$ whenever $x = o + r(\varphi)e(\varphi) \in \partial S$. Let α, β be the direction angles of $p - o$ and $q - o$, and let $\delta = \angle poq \in [0, \pi]$. Let J be the window of directions pointing from o into the chord $[p, q]$; it has angular length δ. Define the closed radial sector

$$W = \{\, o + \rho e(\varphi) : \varphi \in J,\ 0 \le \rho \le r(\varphi) \,\} \subseteq S.$$

W misses U: for $\varphi \in J$ the ray from o in direction $e(\varphi)$ meets $[p, q]$ at a point x; the segment $[o, x]$ lies in the convex set B (both endpoints do), which is disjoint from U; beyond x the ray lies strictly on the far side of the line pq, while U lies on the near side. Every point of W therefore lies in $S \cap C$ or in a filled component, so

$$\operatorname{area}(U) \;=\; 1 - \operatorname{area}(S \cap C) - \sum_{U' \text{ filled}} \operatorname{area}(U') \;\le\; 1 - \operatorname{area}(W).$$

For the arc length: for a convex body M, each coordinate increases along one of the two boundary arcs between its extremes and decreases along the other, so $P_1(\partial M) = 2w_x(M) + 2w_y(M)$; in particular $P_1(\partial\,\cdot\,)$ is monotone under inclusion of convex sets. Since o lies strictly off the line pq, the set $\operatorname{conv}\{p, o, q\}$ is a nondegenerate triangle contained in B, and

$$P_1(\Gamma) + |p - q|_1 \;=\; P_1(\partial B) \;\ge\; P_1\bigl(\partial \operatorname{conv}\{p, o, q\}\bigr) \;=\; |p - o|_1 + |o - q|_1 + |p - q|_1.$$

Since $p, q \in \partial S$, this gives

$$P_1(\Gamma) \;\ge\; |p - o|_1 + |o - q|_1 \;=\; r(\alpha)\, m(\alpha) + r(\beta)\, m(\beta). \tag{6}$$

For the turning: ∂B is a closed convex curve of total turning 2π, consisting of Γ, the straight chord, and corners at p and q with exterior angles $\varepsilon_p, \varepsilon_q$. The turning along the relative interior of Γ equals $\kappa_C(\Gamma \cap \operatorname{int} S)$: the vertices of C interior to Γ carry the same exterior angles in B as in C, and p, q are not vertices of C by transversality. From $\operatorname{conv}\{p, o, q\} \subseteq B$, the interior angle of B at p is at least $\angle opq$, so $\varepsilon_p \le \pi - \angle opq$; similarly $\varepsilon_q \le \pi - \angle oqp$. Since $\angle opq + \angle oqp = \pi - \delta$,

$$\kappa_C(\Gamma \cap \operatorname{int} S) \;=\; 2\pi - \varepsilon_p - \varepsilon_q \;\ge\; \angle opq + \angle oqp \;=\; \pi - \delta. \tag{7}$$

Step 4: the wedge inequality. By the sector bound $\operatorname{area}(U) \le 1 - \operatorname{area}(W)$ together with (6) and (7), the wrapping component obeys $\operatorname{area}(U) \le \frac{1}{2}P_1(\Gamma) + \kappa_C(\Gamma \cap \operatorname{int} S)/(2\pi)$ once we establish

$$\operatorname{area}(W) + \tfrac{1}{2}\big(r(\alpha)\, m(\alpha) + r(\beta)\, m(\beta)\big) + \frac{\pi - \delta}{2\pi} \;\ge\; 1. \tag{8}$$

Write $\delta = j\pi/2 + u$ with $j \in \{0, 1, 2\}$ and $u \in [0, \pi/2]$, taking $u = 0$ if $j = 2$. First, $\operatorname{area}(W) = \frac{1}{2}\int_J r(\varphi)^2\, d\varphi$, and r^2 is $\pi/2$-periodic, each full period contributing $\frac{1}{2}\int r^2 = \frac{1}{4}$, one quarter of $\operatorname{area}(S)$. Among windows I of length $u \le \pi/2$, the integral $\frac{1}{2}\int_I r^2$ is smallest when I is centered on a side normal of S, since r^2 is even about each side normal and increasing in the angular distance from it; there $r(\varphi) = 1/(2\cos\varphi)$ for $|\varphi| \le \pi/4$, and

$$\tfrac{1}{2}\int_{-u/2}^{u/2} \frac{d\varphi}{4\cos^2\varphi} \;=\; \tfrac{1}{4}\tan\frac{u}{2}.$$

Hence $\operatorname{area}(W) \ge j/4 + \frac{1}{4}\tan(u/2)$. Second, $r \ge \frac{1}{2}$, since S contains the disk of radius $\frac{1}{2}$ about o; and m is $\pi/2$-periodic and concave on each period with $m(0) = 1$, so $m(\beta) = m(\alpha + u)$, and minimizing over α in one quadrant (both terms are shifted sines; concavity pushes the minimum to the endpoints, where one argument is a multiple of $\pi/2$) gives

$$m(\alpha) + m(\beta) \;\ge\; 1 + \cos u + \sin u.$$

Substituting these bounds and $(\pi - \delta)/(2\pi) = \frac{1}{2} - j/4 - u/(2\pi)$, the left side of (8) minus 1 is at least

$$H(u) \;=\; \tfrac{1}{4}\Big(\tan\frac{u}{2} + \cos u + \sin u - 1\Big) - \frac{u}{2\pi}\,;$$

the j-terms cancel exactly: the sector bound contributes $j/4$ and the turning term $-(j\pi/2)/(2\pi) = -j/4$. Now $H(0) = H(\pi/2) = 0$, and

$$H''(u) \;=\; \tfrac{1}{8}\sec^2\frac{u}{2}\,\tan\frac{u}{2} - \tfrac{1}{4}(\cos u + \sin u) \;\le\; 0\,:$$

with $z = \tan(u/2) \in [0, 1]$, so that $\sec^2(u/2) = 1 + z^2$, $\cos u = (1 - z^2)/(1 + z^2)$, and $\sin u = 2z/(1 + z^2)$, clearing the positive denominators turns $H'' \le 0$ into

$$(1 - z)\big(z^4 + z^3 + 3z^2 + 5z + 2\big) \;\ge\; 0,$$

which holds on $[0, 1]$. A concave function vanishing at both endpoints of $[0, \pi/2]$ is nonnegative there, so $H \ge 0$ and (8) holds.

Now sum over the components of $S \setminus C$. The crossing arcs are distinct, their relative interiors are disjoint subsets of $\partial C \cap \operatorname{int} S$, and arc endpoints carry no P_1-mass; so the charges (5) for the filled components, together with the charge $\frac{1}{2}P_1(\Gamma) + \kappa_C(\Gamma \cap \operatorname{int} S)/(2\pi)$ for the at most one wrapping component and $\kappa_C(\Gamma \cap \operatorname{int} S) \le \kappa_C(\operatorname{int} S)$, give

$$\operatorname{area}(S \setminus C) \;\le\; \tfrac{1}{2}\, P_1(\partial C \cap \operatorname{int} S) + \frac{\kappa_C(\operatorname{int} S)}{2\pi},$$

which is (4) for transverse polygons.

Step 5: general C by centered dilation. Let C be compact convex with $o \in C$ and $\operatorname{int} C \ne \emptyset$. For $\lambda > 1$ set $C_\lambda = o + \lambda(C - o)$; then $C \subseteq C_\lambda$ and $\bigcap_{\lambda > 1} C_\lambda = C$, so $\operatorname{area}(S \setminus C_\lambda)$ increases to $\operatorname{area}(S \setminus C)$ as $\lambda \downarrow 1$.

Fix $\lambda > 1$ and choose transverse convex polygons $P_n \supseteq C_\lambda$ with $P_n \to C_\lambda$ in the Hausdorff metric; outer polygonal approximants exist, and transversality is achieved by a small perturbation of their vertices. Each P_n contains o, so (4) holds for P_n. As $n \to \infty$, $\operatorname{area}(S \setminus P_n) \to \operatorname{area}(S \setminus C_\lambda)$, and the coordinate-variation measures defining P_1 and the turning measures of convex bodies converge weakly under Hausdorff convergence, a standard fact of the theory of convex bodies [11]. Since S is closed, the portmanteau upper bound for closed sets gives, allowing extra mass on ∂S,

$$\limsup_n \Big[\tfrac{1}{2}\, P_1(\partial P_n \cap \operatorname{int} S) + \frac{\kappa_{P_n}(\operatorname{int} S)}{2\pi}\Big] \;\le\; \tfrac{1}{2}\, P_1(\partial C_\lambda \cap S) + \frac{\kappa_{C_\lambda}(S)}{2\pi},$$

whence $\operatorname{area}(S \setminus C_\lambda) \le \frac{1}{2} P_1(\partial C_\lambda \cap S) + \kappa_{C_\lambda}(S)/(2\pi)$.

Rescale. The dilation $x \mapsto o + \lambda(x - o)$ maps C onto C_λ and the contracted square $S^{(\lambda)} = o + \lambda^{-1}(S - o)$ onto S; it scales P_1 by λ and leaves turning invariant, so

$$\tfrac{1}{2}\, P_1(\partial C_\lambda \cap S) + \frac{\kappa_{C_\lambda}(S)}{2\pi} \;=\; \frac{\lambda}{2}\, P_1\big(\partial C \cap S^{(\lambda)}\big) + \frac{\kappa_C(S^{(\lambda)})}{2\pi}.$$

Since o is the center of S, each $S^{(\lambda)}$ is a compact subset of $\operatorname{int} S$, and as $\lambda \downarrow 1$ the sets $S^{(\lambda)}$ increase with union $\operatorname{int} S$. Monotone convergence for the two boundary measures gives $P_1(\partial C \cap S^{(\lambda)}) \to P_1(\partial C \cap \operatorname{int} S)$ and $\kappa_C(S^{(\lambda)}) \to \kappa_C(\operatorname{int} S)$, while $\lambda \to 1$ and $\operatorname{area}(S \setminus C_\lambda) \uparrow \operatorname{area}(S \setminus C)$. Passing to the limit yields (4); no length or turning mass on ∂S enters the bound. $\square$

4.2 Proof of the convex bound

Summing the clipping inequality over a packing proves the theorem.

Proof of Theorem 1.1. Suppose first that $\operatorname{int} K \neq \emptyset$, and let $S_1, \dots, S_N$ be interior-disjoint unit squares with centers in K. Lemma 4.1 with $C = K$, in its equivalent form, gives for each i

$$1 \;\le\; \operatorname{area}(S_i \cap K) + \tfrac{1}{2}\, P_1(\partial K \cap \operatorname{int} S_i) + \frac{\kappa_K(\operatorname{int} S_i)}{2\pi}.$$

Sum over i. Three different measures appear: area on K, ℓ^1 length on ∂K, and turning on ∂K, with total masses $\operatorname{area}(K)$, $P_1(\partial K)$, and 2π. Each is evaluated on the open sets $\operatorname{int} S_i$ (for the first, $\operatorname{area}(S_i \cap K) = \operatorname{area}(\operatorname{int} S_i \cap K)$ because ∂S_i is Lebesgue-null), and these sets are pairwise disjoint, so each of the three sums is at most the corresponding total mass:

$$N \;\le\; \operatorname{area}(K) + \tfrac{1}{2}\, P_1(\partial K) + \frac{\kappa_K(\partial K)}{2\pi} \;=\; \operatorname{area}(K) + \tfrac{1}{2}\, P_1(\partial K) + 1.$$

Around the closed convex curve ∂K each coordinate increases along one of the two arcs between its extremes and decreases along the other, so its total variation is twice the corresponding width, and $P_1(\partial K) = 2w_x(K) + 2w_y(K)$. Hence $N \le \operatorname{area}(K) + w_x(K) + w_y(K) + 1$.

For the Minkowski identity, write $Q_0 = I_x \oplus I_y$ with $I_x = [-\frac{1}{2}, \frac{1}{2}] \times \{0\}$ and $I_y = \{0\} \times [-\frac{1}{2}, \frac{1}{2}]$. Summing K with I_x lengthens every nonempty horizontal section by exactly 1, and the sections are nonempty over a set of heights of measure $w_y(K)$, so $\operatorname{area}(K \oplus I_x) = \operatorname{area}(K) + w_y(K)$ by Cavalieri. Summing with I_y lengthens every nonempty vertical section of $K \oplus I_x$ by 1 over a set of abscissae of measure $w_x(K \oplus I_x) = w_x(K) + 1$, so

$$\operatorname{area}(K \oplus Q_0) \;=\; \operatorname{area}(K) + w_y(K) + \big(w_x(K) + 1\big) \;=\; \operatorname{area}(K) + w_x(K) + w_y(K) + 1,$$

matching the bound.

If $\operatorname{int} K = \emptyset$, apply the case above to the outer parallel bodies $K_\varepsilon = K \oplus D(0, \varepsilon)$, $\varepsilon > 0$: the centers lie in $K \subseteq K_\varepsilon$, so $N \le \operatorname{area}(K_\varepsilon) + w_x(K_\varepsilon) + w_y(K_\varepsilon) + 1$, and the right side is continuous as $\varepsilon \downarrow 0$ with limit $\operatorname{area}(K) + w_x(K) + w_y(K) + 1$. $\square$

Remark 4.2 (The score is the clipping budget). For $K = [0, a] \times [0, b]$ the theorem recovers Theorem 1.2 term for term. Along the axis-parallel boundary ∂K the ℓ^1 length coincides with ordinary length, so $\frac{1}{2} P_1(\partial K \cap \operatorname{int} S)$ is the edge term of the score (3) at weight $\frac{1}{2}$; each corner of the rectangle carries turning $\pi/2$, hence normalized mass $(\pi/2)/(2\pi) = \frac{1}{4}$, the corner weight. The totals $\frac{1}{2} P_1(\partial K) = a + b$ and $\kappa_K(\partial K)/(2\pi) = 1$ reproduce the budget $ab + a + b + 1 = (a+1)(b+1) = \operatorname{area}(K \oplus Q_0)$. The clipping lemma restricted to rectangles is precisely Lemma 3.2.

Remark 4.3 (Sharpness). For $K = [0, a] \times [0, b]$ with $a, b \in \mathbb{N}$ the bound is attained, by Theorem 1.2. For generic K it is not: a point K gives the bound 1, which is attained, while a disk of small radius gives a bound slightly above 1 although a disk of diameter less than 1 holds at most one center. Theorem 1.1 is thus the exact extension of the integer-rectangle phenomenon to arbitrary convex center regions, such as tilted strips. For those it supplies pruning bounds in computer searches that Theorem 1.2 does not reach.

5 Narrow strips: the very-narrow law and three-center obstructions

This section proves Theorem 1.3, develops two lemmas about the rounded neighbourhood $Q \oplus D(0, \frac{1}{2})$ of a unit square, and uses them to prove three positional obstructions, the assembly theorem (Theorem 5.10), and Corollary 1.4. Throughout, w denotes the strip width and $d = \sqrt{1 - w^2}$.

5.1 The very-narrow law

Proposition 5.1 (Ordered strip bound). *For $0 \le w < 1$ and $h \ge 0$, $N_{\max}(w, h) \le \lfloor h/d \rfloor + 1$. Consequently, if $n - 1 \le h < nd$ then $N_{\max}(w, h) = n$.*

Proof. Order the centers by height. Consecutive centers differ by at most w horizontally, hence by at least d vertically by Corollary 2.2; with n centers, $h \ge (n - 1)d$. For the second claim, $h < nd$ forbids $n + 1$ centers, while $h \ge n - 1$ admits the vertical chain of n axis-parallel unit squares centered at unit spacing on a vertical line, interior-disjoint by Lemma 2.3. □

The diagonal chain below is the lower-bound construction for Theorem 1.3; it is stated as a lemma because Corollary 1.4 and the three-center threshold constructions of the next section reuse it.

Lemma 5.2 (Diagonal chain). *For all $w, h \ge 0$, $N_{\max}(w, h) \ge \left\lfloor \sqrt{w^2 + h^2} \right\rfloor + 1$.*

Proof. Let $D = \sqrt{w^2 + h^2}$. If $D = 0$ take one square. Otherwise let $e = (w, h)/D$ and give every square a side direction parallel to e. Place centers at ke for $0 \le k \le \lfloor D \rfloor$; they lie on the segment from $(0, 0)$ to (w, h), hence in $[0, w] \times [0, h]$. Any two of the squares have common side axes and centers separated by $|k - k'| \ge 1$ along the axis e, so they are interior-disjoint by Lemma 2.3; consecutive squares share a full edge. □

Lemma 5.3 (Short-offset chord). *A line at distance at most $\tau = (\sqrt{2} - 1)/2$ from the center of a unit square meets it in a segment of length at least 1.*

Proof. Rotate so that the line is vertical at abscissa x from the center, with $0 \le x \le \tau$, and the square makes angle $\theta \in [0, \pi/4]$ with the axes; write $c = \cos\theta$, $s = \sin\theta$.

If $x \le (c - s)/2$, the line crosses the pair of opposite sides of the square nearest to horizontal and the chord has length $1/c \ge 1$.

Otherwise $s > 0$ (at $\theta = 0$ the first case covers all $x \le \tau < 1/2$), and the chord cuts a corner, with length $\big((c + s)/2 - x\big)/(sc)$. Put $u = c + s \in [1, \sqrt{2}]$, so $sc = (u^2 - 1)/2$. The function $u \mapsto (u - u^2 + 1)/2$ is decreasing on $[1, \sqrt{2}]$, with minimum $(\sqrt{2} - 1)/2 = \tau$ at $u = \sqrt{2}$; hence

$$\frac{c + s}{2} - sc \;=\; \frac{u - u^2 + 1}{2} \;\ge\; \tau \;\ge\; x,$$

which rearranges to $\big((c + s)/2 - x\big)/(sc) \ge 1$. □

Proof of Theorem 1.3. Let $D = \sqrt{w^2+h^2}$. Lemma 5.2 gives $N_{\max}(w,h) \geq \lfloor D \rfloor + 1$. For the upper bound, suppose $n \geq 2$ interior-disjoint unit squares $S_0, \dots, S_{n-1}$ have centers $P_0, \dots, P_{n-1} \in [0,w] \times [0,h]$, ordered so that $y_0 \leq \cdots \leq y_{n-1}$; write $c = |P_{n-1} - P_0|$ and $Y = y_{n-1} - y_0$. Since $P_0, P_{n-1} \in [0,w] \times [0,h]$ we have $c \leq D$, so it suffices to prove $c \geq n-1$. For $n = 2$ this is Lemma 2.1, so assume $n \geq 3$ and let ℓ be the line through P_0 and P_{n-1}. The plan: each middle center lies within distance τ of ℓ, so its square cuts a chord of length at least 1 on ℓ, and these chords pack disjointly between two half-chords of the end squares.

Step 1 (feet on the segment). By Corollary 2.2 consecutive vertical gaps are at least d, so for $0 < i < n-1$ we have $y_i - y_0 \geq d$ and $Y \geq 2d$. Horizontal coordinates differ by at most w, so

$$(P_i - P_0) \cdot (P_{n-1} - P_0) \;\geq\; (y_i - y_0)\,Y - w^2 \;\geq\; 2d^2 - w^2 \;=\; 2 - 3w^2 \;>\; 0,$$

using $w \leq \tau < 1/\sqrt{3}$, and symmetrically $(P_{n-1} - P_i) \cdot (P_{n-1} - P_0) > 0$. Hence the perpendicular foot of each middle center on ℓ lies on the segment $[P_0, P_{n-1}]$.

Step 2 (offsets at most τ). The triangle $P_0 P_i P_{n-1}$ lies in a $w \times Y$ rectangle, so its area is at most $wY/2$, and the distance r_i from P_i to ℓ satisfies $r_i \leq wY/c \leq w \leq \tau$, using $c \geq Y$.

Step 3 (chords). The line ℓ passes through the centers of the two end squares; each contains its incircle, so each meets ℓ in a chord through its center whose half-chord pointing toward the other end square has length at least $1/2$. Each middle square S_i meets ℓ in a chord of length at least 1 by Lemma 5.3, since $r_i \leq \tau$. As $r_i < 1/2$, this chord passes through the interior of S_i and contains the perpendicular foot of P_i (the foot lies in the incircle of S_i), hence a point of $[P_0, P_{n-1}]$.

Step 4 (disjointness of the chords). For every square of the family, ℓ lies at distance less than $1/2$ from its center ($r_i \leq \tau < 1/2$ for the middle squares, 0 for the end squares), whereas a line containing a side of a unit square lies at distance exactly $1/2$ from its center. So ℓ contains no side of any of the squares, and the intersection of ℓ with each square is a chord whose relative interior lies in the open square. The open squares are pairwise disjoint, so the relative interiors of the n chords are pairwise disjoint.

Step 5 (packing). Each middle chord is connected, its relative interior is disjoint from the two inward half-chords, and it contains a point of $[P_0, P_{n-1}]$; hence it lies between the inner endpoints of the two half-chords. Summing lengths along ℓ,

$$c \;\geq\; \tfrac{1}{2} + (n-2) \cdot 1 + \tfrac{1}{2} \;=\; n - 1.$$

Thus $n - 1 \leq c \leq D$, so $N_{\max}(w,h) \leq \lfloor D \rfloor + 1$. ☐

Remark 5.4 (Sharpness in w). For every $\xi < w < 1$ the conclusion of Theorem 1.3 fails at h just below $\sqrt{4 - w^2}$: there $\lfloor D \rfloor + 1 = 2$, yet three centers fit, at height $2\sqrt{2} - 2w < \sqrt{4 - w^2}$ when $w \leq 1/\sqrt{2}$ and at height $1/w < \sqrt{4 - w^2}$ when $w > 1/\sqrt{2}$ (Theorem 1.5). On the bounded range $D < 2$ the law nevertheless extends to all $w \leq \xi$: this is Corollary 1.4, proved in Section 5.3.

5.2 Rounded neighbourhoods

Write $\nu(x) = \min\{2,\ 2\sqrt{2} - 2x\}$. The two lemmas of this subsection turn interior-disjointness from a unit square Q into metric information about the neighbourhood $K = Q \oplus D(0, \tfrac{1}{2})$: no other center lies in $\operatorname{int} K$, and every line at distance $x \leq 1/\sqrt{2}$ from the center of Q crosses K in a segment of length at least $\nu(x)$.

Lemma 5.5 (Rounded exclusion). *Let Q be a unit square with center O and $K = Q \oplus D(0, \tfrac{1}{2})$. If a unit square Q' is interior-disjoint from Q, then $c(Q') \notin \operatorname{int} K$.*

Proof. For a unit vector u let $h_Q(u) = \sup_{p \in Q} (p - O) \cdot u$, the support radius of Q from O, and likewise $h_{Q'}$ from $P = c(Q')$; by (2), $h_{Q'}(u) \geq 1/2$. The body K has support radius $h_Q(u) + 1/2$

from O. Suppose $P \in \operatorname{int} K$. Since Q is centrally symmetric about O, so is K, and central symmetry licenses the absolute value in

$$|(P-O)\cdot u| \;<\; h_Q(u)+\tfrac{1}{2} \;\le\; h_Q(u)+h_{Q'}(u) \qquad \text{for every unit } u.$$

But Q and Q' are interior-disjoint convex bodies, so some line separates them: there is a unit vector u_0 and $t \in \mathbb{R}$ with $(p-O)\cdot u_0 \le t$ on Q and $(p-O)\cdot u_0 \ge t$ on Q'. Then $t \ge h_Q(u_0)$ and, since Q' is centrally symmetric about P, $(P-O)\cdot u_0 - h_{Q'}(u_0) \ge t$, whence $(P-O)\cdot u_0 \ge h_Q(u_0)+h_{Q'}(u_0)$ — a contradiction. □

Lemma 5.6 (Rounded chord)**.** *Let Q be a unit square centered at the origin and $K = Q \oplus D(0,\frac{1}{2})$. For $0 \le x \le 1/\sqrt{2}$, every line at distance x from the origin meets K in a segment of length at least $\nu(x)$.*

Proof. The plan: parametrize ∂K, classify which pair of boundary pieces the line meets, and reduce each case by concavity to endpoint and kink checks.

Setup. Rotate so that the line is vertical at abscissa $x \ge 0$ and the square makes angle $\theta \in [0,\pi/4]$ with the axes; write $c=\cos\theta$, $s=\sin\theta$, $a=(c-s)/2$, $h_1=(c+s)/2$. The vertices of Q are $\pm A$ and $\pm B$ with $A=(a,h_1)$ and $B=(h_1,-a)$. Since $Q=\operatorname{conv}\{\pm A,\pm B\}$,

$$K \;=\; \operatorname{conv}\bigl(D(A,\tfrac{1}{2}) \cup D(-A,\tfrac{1}{2}) \cup D(B,\tfrac{1}{2}) \cup D(-B,\tfrac{1}{2})\bigr),$$

so ∂K consists of four circular arcs and the four tangent segments obtained by translating the sides of Q outward by $1/2$. The vertical section of K at abscissa $x \ge 0$ runs between an upper and a lower profile, whose consecutive pieces are: for the upper profile, the shifted top side $y=(1+sx)/c$ for $x \le a-s/2$, the arc of $D(A,\frac{1}{2})$ for $a-s/2 \le x \le a+c/2$, and the shifted upper-right side $y=(1-cx)/s$ beyond; for the lower profile, the arc of $D(-A,\frac{1}{2})$ for $x \le s/2-a$ (nonempty only if $c \le 2s$), the shifted bottom side $y=(sx-1)/c$ for $s/2-a \le x \le h_1+s/2$, and the arc of $D(B,\frac{1}{2})$ beyond.

Case list. For $x \le 1/\sqrt{2}$ the possible (upper, lower) profile pairs are exactly (top, bottom), (arc A, arc $-A$), (arc A, bottom), (upper-right, bottom) and (arc A, arc B); the fourth occurs only when $c \le 3s$, the fifth only when $c \ge 3s$, and the pair (upper-right, arc B) never occurs in this range, its two range conditions being incompatible.

Reduction. Let $V(x)$ be the length of the vertical section of K at x. On each case interval V is concave, being a sum of concave arc terms and linear side terms; ν is concave and linear on each side of its kink at $x=\sqrt{2}-1$. Hence on each case interval it suffices to verify $V \ge \nu$ at the endpoints and, when it lies in the interval, at the kink $x=\sqrt{2}-1$. We use this reduction in all five cases.

Case (top, bottom). $V = 2/c \ge 2 \ge \nu$.

Case (upper-right, bottom). Here $V=(1-cx)/s+(1-sx)/c=(c+s-x)/(cs)$. Put $u=c+s \in [1,\sqrt{2}]$, so $2cs=u^2-1$. Two direct checks cover the whole range $0 \le x \le 1/\sqrt{2}$. (i) $V \ge 2\sqrt{2}-2x$: after clearing denominators the coefficient $1-2cs$ of x is nonnegative, so the worst point is $x=1/\sqrt{2}$, where the claim reduces to $\sqrt{2}\,u-(u^2-1)\ge 1$, i.e. $u(\sqrt{2}-u)\ge 0$. (ii) For $x \le \sqrt{2}-1$, $V \ge 2$ reduces to $x \le u-(u^2-1)$, whose right side is decreasing in u with minimum $\sqrt{2}-1$ at $u=\sqrt{2}$.

Case (arc A, arc $-A$). This pair requires $x \le s/2-a$, hence $c \le 2s$ and $x \le s/2 < \sqrt{2}-1$, so $\nu=2$ on the case interval. From $c \le 2s$, $\sin 2\theta \ge 4/5$, so $u^2 = 1+\sin 2\theta \ge 9/5 > 25/16$ and $u \ge 5/4$. At the left endpoint $x=0$, $V=u+\sqrt{u^2-1} \ge \frac{5}{4}+\frac{3}{4}=2$. At the right endpoint the pair degenerates into (arc A, bottom), treated next.

Case (arc A, bottom). Here

$$V=g(x):=h_1+\sqrt{\tfrac{1}{4}-(x-a)^2}+\frac{1-sx}{c} \quad \text{on} \quad \max\{a-\tfrac{s}{2},\,\tfrac{s}{2}-a\} \;\le\; x \;\le\; \min\{a+\tfrac{c}{2},\,h_1+\tfrac{s}{2}\}.$$

At the left endpoint the value of V agrees, by continuity of the profiles, with its value in the adjacent case — (top, bottom) if $c \geq 2s$, (arc A, arc $-A$) if $c \leq 2s$ — where the bound is proved; at the right endpoint likewise with (upper-right, bottom) ($c \leq 3s$) or (arc A, arc B) ($c \geq 3s$). It remains to check the kink: whenever $\sqrt{2}-1$ lies in the case interval, we must verify $g(\sqrt{2}-1) \geq 2$. At the extreme angles of the θ-range in which this happens, the inequality degenerates to $2/c \geq 2$ and to $(3-4c)/(2c) \geq 0$, the latter at the angle θ_1 defined by $\sqrt{2}-1 = a + c/2$, where $c = \cos\theta_1 = 0.7468\ldots < 3/4$. In the interior of the θ-range the inequality $g(\sqrt{2}-1) \geq 2$ is verified numerically — the margin exceeds 0.007 on a dense grid of angles, and 0.008 away from the endpoint angles — and it is proved symbolically only at the two endpoint angles; no fully symbolic proof of the interior range is known.

Case (arc A, arc B). Here $V = c + \sqrt{\frac{1}{4} - (x-a)^2} + \sqrt{\frac{1}{4} - (x-h_1)^2}$ on $h_1 + s/2 \leq x \leq 1/\sqrt{2}$; the interval is nonempty only when $c + 2s \leq \sqrt{2}$, i.e. $s \leq (2\sqrt{2}-\sqrt{3})/5 = \xi/2$, which forces $c \geq 0.975$. On the interval $x \geq 1/2 > \sqrt{2}-1$, so $\nu = 2\sqrt{2} - 2x$ and we check the two endpoints. At the left endpoint $x = h_1 + s/2$,

$$V + 2x \;=\; \tfrac{5}{2}c + 2s + \sqrt{\tfrac{1}{4} - \tfrac{9}{4}s^2},$$

which is increasing in s — its derivative exceeds $2 - \frac{5s}{2c} - \frac{9s/4}{\sqrt{1/4 - 9s^2/4}} > 0$ on the range — with value $3 > 2\sqrt{2}$ at $s = 0$. At the right endpoint $x = 1/\sqrt{2}$ the two square roots are at least 0.37 and 0.45 respectively and $c \geq 0.975$, giving $V + 2x \;\geq\; 0.975 + 0.37 + 0.45 + \sqrt{2} \;>\; 2\sqrt{2}$. (These decimal constants are rigorous but coarse interval bounds.)

In every case $V(x) \geq \nu(x)$. □

The bound ν is exact: at $\theta = \pi/4$ the section length equals $2\sqrt{2} - 2x$ for $1/(2\sqrt{2}) \leq x \leq 1/\sqrt{2}$, and at $\theta = 0$ it equals 2 for $x \leq 1/2$.

5.3 Three-center obstructions

Three obstructions show that a narrow rectangle holding three centers must be large; a corner-counting assembly then yields Theorem 5.10, from which Corollary 1.4 follows.

Proposition 5.7 (Same-side obstruction)**.** *Let $0 \leq w \leq 1/\sqrt{2}$. If three interior-disjoint unit squares have centers $A = (0,0)$, $C = (0,h)$ and $B = (x,y)$ with $0 \leq x \leq w$ and $0 \leq y \leq h$, then $h \geq \min\{2,\ 2\sqrt{2} - 2w\} = \nu(w)$.*

Proof. Let Q_B be the square centered at B and $K_B = Q_B \oplus D(0, \frac{1}{2})$. By Lemma 5.5 neither A nor C lies in $\operatorname{int} K_B$. The vertical line ℓ through A and C is at distance $x \leq w \leq 1/\sqrt{2}$ from B, so by Lemma 5.6 it meets K_B in a segment σ of length at least $\nu(x) \geq \nu(w)$, as ν is nonincreasing. The point $(0,y)$ lies in K_B: the horizontal chord of Q_B through B has half-length at least 1/2, so the distance from $(0,y)$ to Q_B is at most $\max\{0,\ x - \frac{1}{2}\} \leq 1/\sqrt{2} - \frac{1}{2} < \frac{1}{2}$. Since $K_B \supseteq D(B,1)$ and $x < 1$, the line ℓ meets $\operatorname{int} K_B$, so the relative interior of σ lies in $\operatorname{int} K_B$ and contains neither A nor C. Thus A and C lie on ℓ, outside the relative interior of σ, weakly on opposite sides of the point $(0,y) \in \sigma$; hence $\sigma \subseteq [A,C]$ and $h = |AC| \geq \nu(w)$. □

Lemma 5.8 (Staircase obstruction)**.** *Let $0 \leq w \leq 1/\sqrt{2}$ and $w^2 + h^2 < 4$. If three interior-disjoint unit squares have centers $A = (0,y)$, $B = (x,0)$, $C = (w,h)$ with $0 \leq x \leq w$ and $0 \leq y \leq h$, then $h + 2w \geq 2\sqrt{2}$.*

Proof. Put $t = h - y$, $u = w - x$, $c = |BC| = \sqrt{h^2 + u^2}$, and let r be the distance from A to the line BC. We prove $c + 2r \leq h + 2w$ and $c + 2r \geq 2\sqrt{2}$.

From $|AC| \geq 1$ (Lemma 2.1) with $A - C = -(w, t)$ we get $w^2 + t^2 \geq 1$, so $t \geq d$; also $u \leq w \leq d$, since $w \leq 1/\sqrt{2}$. Twice the area of the triangle ABC is $xt + wy = wh - ut \leq wh - ud$, so $r \leq (wh - ud)/c$. Define

$$F(u) = (h + 2w)\sqrt{h^2 + u^2} - (h^2 + u^2) - 2wh + 2ud.$$

Then $F(0) = 0$ and

$$F'(u) = \frac{(h + 2w)\, u}{\sqrt{h^2 + u^2}} - 2u + 2d \;\geq\; 2(d - u) \;\geq\; 0,$$

using $u \leq d$; hence $F(u) \geq 0$, which rearranges to

$$c + 2r \;\leq\; c + \frac{2(wh - ud)}{c} \;\leq\; h + 2w. \tag{9}$$

The foot of the perpendicular from A to the line BC lies on the segment $[B, C]$:

$$(A - C) \cdot (B - C) = wu + th \;\geq\; 0,$$
$$(A - B) \cdot (C - B) = hy - xu \;\geq\; y^2 - xu \;\geq\; 1 - x^2 - xu \;\geq\; 1 - 2w^2 \;\geq\; 0,$$

using $h \geq y$, $|AB| \geq 1$ and $x, u \leq w \leq 1/\sqrt{2}$.

Moreover $r \leq wh/c \leq w \leq 1/\sqrt{2}$, since $c \geq h$. Let K_A be the rounded neighbourhood of the square centered at A. By Lemma 5.5 neither B nor C lies in $\operatorname{int} K_A$, and by Lemma 5.6 — applicable since $r \leq 1/\sqrt{2}$ — the line BC meets K_A in a segment of length at least $\nu(r)$; the segment contains the foot, which lies in $D(A, 1) \subseteq K_A$. As in Proposition 5.7, B and C lie on the line, outside the relative interior of the segment, weakly on opposite sides of the foot; hence $c = |BC| \geq \nu(r)$. Since $c \leq \sqrt{h^2 + w^2} < 2$, the branch $\nu(r) = 2$ is impossible, so $c \geq 2\sqrt{2} - 2r$, i.e.

$$c + 2r \;\geq\; 2\sqrt{2}. \tag{10}$$

Combining (9) and (10) gives $h + 2w \geq 2\sqrt{2}$. □

Lemma 5.9 (Opposite-corner obstruction)**.** *Let $0 \leq w \leq 1/\sqrt{2}$ and $w^2 + h^2 < 4$. If three interior-disjoint unit squares have centers at $O = (0, 0)$, $C = (w, h)$ and a third point $P = (x, y)$ of $[0, w] \times [0, h]$, then $h + 2w \geq 2\sqrt{2}$.*

Proof. Write $D = \sqrt{w^2 + h^2} < 2$ and let r be the distance from P to the line OC. We may assume $hx \geq wy$: otherwise apply the point reflection through the center of the rectangle, which swaps O and C, keeps P in the rectangle, and replaces $hx - wy$ by its negative. Then $r = (hx - wy)/D$.

From $|OP| \geq 1$ and $y \geq 0$ we get $y \geq \sqrt{1 - x^2}$, so $hx - wy \leq hx - w\sqrt{1 - x^2}$; the right side is increasing in x on $[0, w]$, so $hx - wy \leq w(h - d)$ and $r \leq w(h - d)/D$. The case $u = w$ of the staircase computation ($F(w) \geq 0$, valid since $u \leq d$ throughout $[0, w]$) gives

$$D + 2r \;\leq\; D + \frac{2w(h - d)}{D} \;\leq\; h + 2w.$$

The foot of the perpendicular from P to the line OC lies on the segment $[O, C]$: $(P - O) \cdot (C - O) = xw + yh \geq 0$ and $(P - C) \cdot (O - C) = w(w - x) + h(h - y) \geq 0$ for $P \in [0, w] \times [0, h]$. Moreover $r \leq w(h - d)/D \leq wh/D \leq w \leq 1/\sqrt{2}$. As in Lemma 5.8, the rounded neighbourhood of the square centered at P gives $D = |OC| \geq \nu(r)$; since $D < 2$ the branch $\nu(r) = 2$ is impossible, so $D \geq 2\sqrt{2} - 2r$, i.e. $D + 2r \geq 2\sqrt{2}$. Combining the two displays gives $h + 2w \geq 2\sqrt{2}$. □

Theorem 5.10 (Narrow three-center obstruction)**.** *Let $0 \leq w \leq 1/\sqrt{2}$. If $w^2 + h^2 < 4$ and $h + 2w < 2\sqrt{2}$, then $N_{\max}(w, h) \leq 2$.*

Proof. Suppose three interior-disjoint unit squares have centers in $[0, w] \times [0, h]$. Replace the rectangle by the axis-parallel bounding box of the three centers and translate it to $[0, w'] \times [0, h']$: then $w' \le w$ and $h' \le h$, so $w' \le 1/\sqrt{2}$, $w'^2 + h'^2 < 4$ and $h' + 2w' \le h + 2w < 2\sqrt{2}$ — the strict hypotheses persist — and every side of $[0, w'] \times [0, h']$ contains a center. Each case below ends in a contradiction.

Since $w' \le 1/\sqrt{2} < 1$, no horizontal side contains two centers: they would be at distance at most $w' < 1$, contradicting Lemma 2.1.

Suppose some vertical side contains centers at both of its endpoints. After a reflection the centers include $(0, 0)$ and $(0, h')$, and the third lies in $[0, w'] \times [0, h']$, so Proposition 5.7 gives $h' \ge \min\{2, 2\sqrt{2} - 2w'\}$; but $h' < 2$ (from $h'^2 < 4$) and $h' < 2\sqrt{2} - 2w'$ — a contradiction. (If $w' = 0$, all three centers lie on one vertical side and its endpoints are centers, so this case applies.)

Otherwise, a center not at a corner of the box lies on at most one side, and three centers cover four sides, so at least one center lies at a corner. If two centers lie at corners, they lie either on a common vertical side — the previous case — or at opposite corners, since two corners on a common horizontal side are excluded by $w' < 1$; after a reflection the centers include $(0, 0)$ and (w', h'), and Lemma 5.9 gives $h' + 2w' \ge 2\sqrt{2}$, contradicting $h' + 2w' < 2\sqrt{2}$. If exactly one center lies at a corner, reflect so that it is (w', h'); the remaining two centers must cover the sides $x = 0$ and $y = 0$, one each, giving centers $(0, y)$, $(x, 0)$, (w', h') with $0 \le x \le w'$, $0 \le y \le h'$ — the configuration of Lemma 5.8, whence $h' + 2w' \ge 2\sqrt{2}$, again a contradiction. □

Proof of Corollary 1.4. Lemma 5.2 gives $N_{\max}(w, h) \ge \lfloor D \rfloor + 1$. If $D < 1$, any two centers in $[0, w] \times [0, h]$ would be at distance at most $D < 1$, contradicting Lemma 2.1, so $N_{\max}(w, h) = 1 = \lfloor D \rfloor + 1$. Let $1 \le D < 2$; it remains to show $N_{\max}(w, h) \le 2$. The constant $\xi = (4\sqrt{2} - 2\sqrt{3})/5$ is the smaller positive root of $5w^2 - 8\sqrt{2}\, w + 4 = 0$ (the roots are $(4\sqrt{2} \pm 2\sqrt{3})/5$), equivalently the smaller root of $\sqrt{4 - w^2} + 2w = 2\sqrt{2}$; the function $w \mapsto \sqrt{4 - w^2} + 2w$ is increasing on $[0, \xi]$, so $w \le \xi$ gives $\sqrt{4 - w^2} + 2w \le 2\sqrt{2}$. Since $D < 2$ we have $h < \sqrt{4 - w^2}$, hence $h + 2w < 2\sqrt{2}$; and $\xi < 1/\sqrt{2}$. Theorem 5.10 yields $N_{\max}(w, h) \le 2$. □

6 The three-center threshold and exact wide bands

This section proves Theorems 1.5 and 1.6. Constructions come first: Section 6.1 builds the two sheared families consumed by the wide threshold of Section 6.2, by the third branch of Theorem 1.5, and by the band lower bounds.

6.1 Two sheared constructions

Both families place every square in one tilted frame and separate centers along a shared side axis, so that Lemma 2.3 certifies disjointness.

Proposition 6.1 (Two-column odd chain)**.** *Let $0 < w \le 1$ and let $r \ge 1$ be an integer. If $h \ge r/w$ then $N_{\max}(w, h) \ge 2r + 1$.*

Proof. Let $s = \sqrt{1 - w^2}$ and give every square the orthonormal side directions $u = (w, s)$, $v = (-s, w)$. Place $r + 1$ left-column centers $L_k = (0,\, k/w)$ for $0 \le k \le r$ and r right-column centers $R_k = (w,\, s + k/w)$ for $0 \le k \le r - 1$. All $2r + 1$ centers lie in $[0, w] \times [0, r/w]$: the left column tops out at r/w and the right at $s + (r - 1)/w \le r/w$, since $sw \le 1$.

By Lemma 2.3 it suffices to separate each pair of centers by at least 1 along u or v. A same-column difference is $(0, m/w)$ with $1 \le |m| \le r$, and $|(0, m/w) \cdot v| = |m| \ge 1$. A cross-column difference is $R_l - L_k = (w,\, s + m/w)$ with $m = l - k$. If $m = 0$, its projection on u is $w^2 + s^2 = 1$. If $m > 0$, its projection on u is $1 + ms/w \ge 1$. If $m < 0$, its projection on v is $-sw + (s + m/w)w = m$, of absolute value at least 1. Hence the $2r + 1$ squares are interior-disjoint and $N_{\max}(w, h) \ge 2r + 1$. □

Proposition 6.2 (Two-row sheared grid)**.** *Let $0 < s \le 1$ and let $q \ge 1$ be an integer. Then*

$$N_{\max}\Big(\frac{q-1}{s} + \sqrt{1-s^2},\ s\Big) \ \ge\ 2q.$$

In particular, for $0 < w \le 1$ and every integer $r \ge 0$,

$$N_{\max}\Big(w,\ \frac{r}{w} + \sqrt{1-w^2}\Big) \ \ge\ 2r+2.$$

Proof. Let $\gamma = \sqrt{1-s^2}$, $u = (\gamma, s)$, $v = (-s, \gamma)$ and $H = (1/s,\ 0)$, so that u, v are orthonormal and $H \cdot v = -1$. Give every square the side directions u, v and place the $2q$ centers $ku + mH$ for $k \in \{0, 1\}$, $0 \le m \le q-1$. Their horizontal coordinates $k\gamma + m/s$ lie in $[0,\ \gamma + (q-1)/s]$ and their vertical coordinates ks in $[0, s]$.

A difference of distinct centers is $Ku + MH$ with $K \in \{0, \pm 1\}$, $|M| \le q-1$ and $(K, M) \ne (0, 0)$. If $M = 0$ then $|(Ku) \cdot u| = |K| = 1$. If $M \ne 0$ then, since $u \cdot v = 0$ and $H \cdot v = -1$, $|(Ku + MH) \cdot v| = |M| \ge 1$. Each pair of centers is thus separated by at least 1 along a common side axis, and Lemma 2.3 gives the first bound. Taking $s = w$ and $q = r+1$ yields $N_{\max}(r/w + \sqrt{1-w^2},\ w) \ge 2r+2$; the symmetry $N_{\max}(a, b) = N_{\max}(b, a)$ yields the second. □

6.2 The wide threshold

The upper bound in the wide range $1/\sqrt{2} \le w < 1$ rests on one imported result.

Theorem 6.3 (Center-area lemma [7])**.** *If three interior-disjoint unit squares have centers P_1, P_2, P_3 and the triangle $P_1P_2P_3$ is non-obtuse, then its area is at least $1/2$.*

The companion paper [7] proves this by analysing an area-minimising configuration through the reciprocal triangle of its contact forces, and carries a complete Lean formalization of the result. We use it as a black box.

Lemma 6.4 (Obtuse triples in a wide strip)**.** *Let $1/\sqrt{2} \le w < 1$ and let A, B, C be points of a vertical strip of width w with pairwise distances at least 1 and $(A - B) \cdot (C - B) \le 0$ (the triangle ABC is right or obtuse at B; degenerate triples included). Then the vertical span of $\{A, B, C\}$, the difference between the largest and smallest ordinate, is at least $1/w$.*

Proof. We show that the vertical deviations of A and C from B have opposite signs and sum to at least $1/w$. Write $u = A - B = (\alpha, p)$ and $v = C - B = (\beta, q)$, so $u \cdot v \le 0$ and $|u|, |v| \ge 1$; membership in the strip gives $|\alpha|, |\beta|, |\alpha - \beta| \le w$. Put $a = |\alpha|$, $b = |\beta|$, $P = |p|$, $Q = |q|$ and $d = \sqrt{1-w^2}$. From $\alpha^2 + p^2 \ge 1$ and $a \le w$ we get $P \ge \sqrt{1-a^2} \ge d > 0$, and likewise $Q \ge \sqrt{1-b^2} \ge d$.

Step 1: p and q have opposite signs. Suppose $pq > 0$. Then $u \cdot v = \alpha\beta + pq \le 0$ forces $\alpha\beta \le -pq < 0$, so α and β have opposite signs, $a + b = |\alpha - \beta| \le w < 1$, and $pq \le -\alpha\beta = ab$. But then

$$\sqrt{1-a^2}\,\sqrt{1-b^2} \ \le\ PQ \ =\ pq \ \le\ ab,$$

and squaring gives $a^2 + b^2 \ge 1$, contradicting $a^2 + b^2 \le (a+b)^2 < 1$.

So A and C lie on opposite sides of the horizontal line through B, and the vertical span is at least $P + Q$. It remains to prove $P + Q \ge 1/w$; we split on the sign pattern of α, β.

Step 2: $\alpha\beta \le 0$. Then $a + b = |\alpha - \beta| \le w$. By concavity of $t \mapsto \sqrt{1-t^2}$, the minimum of $\sqrt{1-a^2} + \sqrt{1-b^2}$ over $a, b \ge 0$ with $a + b \le w$ is attained at $(a, b) = (w, 0)$: for fixed sum $a + b = t$ concavity puts the minimum at the endpoint $(t, 0)$, and $1 + \sqrt{1-t^2}$ decreases in t. Hence

$$P + Q \ \ge\ \sqrt{1-a^2} + \sqrt{1-b^2} \ \ge\ 1 + d \ \ge\ \frac{1}{w},$$

the last step being $w(1+d) \ge 1$, equivalently $(1-w)\big(w^2(1+w) - (1-w)\big) \ge 0$, which holds on $[1/\sqrt{2}, 1)$.

Step 3: $\alpha\beta \geq 0$. Now $u \cdot v \leq 0$ gives $PQ = -pq \geq \alpha\beta = ab$. If $a^2 + b^2 \leq 1$, then $P+Q \geq \sqrt{1-a^2} + \sqrt{1-b^2}$, whose minimum over $\{0 \leq a, b \leq w,\ a^2+b^2 \leq 1\}$ is $w+d$, attained at $\{a, b\} = \{w, d\}$: the function decreases in each variable, so the minimum lies either where one variable equals w (forcing the other $\leq d$ and the value $\geq d+w$) or on the arc $a^2+b^2 = 1$ with $a, b \in [d, w]$, where the value is $a+b$ and $(a+b)^2 = 1+2ab \geq 1+2wd = (w+d)^2$ because $t \mapsto t\sqrt{1-t^2}$ attains its minimum on $[d, w]$ at the endpoints. If $a^2+b^2 \geq 1$, then $a, b \in [d, w]$, so $|a-b| \leq w-d$ and

$$(P+Q)^2 \;=\; P^2+Q^2+2PQ \;\geq\; (1-a^2)+(1-b^2)+2ab \;=\; 2-(a-b)^2 \;\geq\; 2-(w-d)^2 \;=\; (w+d)^2,$$

using $w^2+d^2 = 1$. In both cases $P+Q \geq w+d \geq 1/w$, since $w(w+d)-1 = wd-d^2 = d(w-d) \geq 0$ for $w \geq 1/\sqrt{2}$. □

Theorem 6.5 (Wide sheared threshold)**.** *Let $1/\sqrt{2} \leq w < 1$. If $h < 1/w$ then $N_{\max}(w, h) \leq 2$; moreover $N_{\max}(w,\ 1/w) \geq 3$.*

Proof. Suppose three interior-disjoint unit squares have centers in $[0, w] \times [0, h]$ with $h < 1/w$. The centers have pairwise distances at least 1 (Lemma 2.1) and lie in a vertical strip of width w. If some labelling of the centers as A, B, C has $(A-B) \cdot (C-B) \leq 0$, Lemma 6.4 makes their vertical span at least $1/w > h$, impossible for centers in $[0, w] \times [0, h]$. Otherwise every angle of the center triangle is acute, so Theorem 6.3 gives area at least $1/2$, while a triangle contained in $[0, w] \times [0, h]$ has area at most $wh/2 < 1/2$: a contradiction. Hence $N_{\max}(w, h) \leq 2$. Proposition 6.1 with $r = 1$ gives $N_{\max}(w,\ 1/w) \geq 3$. □

6.3 Proof of Theorem 1.5

The three branches of T_3 agree at the crossovers: $\sqrt{4-\xi^2} = 2\sqrt{2} - 2\xi$ by the defining equation of ξ, and $2\sqrt{2} - 2w = \sqrt{2} = 1/w$ at $w = 1/\sqrt{2}$.

Proof of Theorem 1.5. Since $N_{\max}(w, \cdot)$ is nondecreasing, it suffices to show that $h < T_3(w)$ forbids a third center and that $N_{\max}(w,\ T_3(w)) \geq 3$.

Obstructions. Let $h < T_3(w)$. For $w \leq \xi$: then $h < \sqrt{4-w^2}$, so $w^2 + h^2 < 4$; and since $w \mapsto \sqrt{4-w^2} + 2w$ is increasing on $[0, \xi]$ (its derivative $2 - w/\sqrt{4-w^2}$ is positive for $w < 1$) with value $2\sqrt{2}$ at ξ, also $h + 2w < \sqrt{4-w^2} + 2w \leq 2\sqrt{2}$. Theorem 5.10 applies, as $w \leq \xi < 1/\sqrt{2}$, and gives $N_{\max}(w, h) \leq 2$. For $\xi \leq w \leq 1/\sqrt{2}$: then $h + 2w < 2\sqrt{2}$; moreover $2\sqrt{2} - 2w \leq \sqrt{4-w^2}$ on this range, since after squaring this reads $5w^2 - 8\sqrt{2}\,w + 4 \leq 0$, which holds between the roots ξ and $(4\sqrt{2} + 2\sqrt{3})/5 > 1/\sqrt{2}$. Hence $h < \sqrt{4-w^2}$, so $w^2 + h^2 < 4$, and Theorem 5.10 applies again. For $1/\sqrt{2} \leq w < 1$: Theorem 6.5 gives $N_{\max}(w, h) \leq 2$.

Constructions at the threshold. For $w \leq \xi$ and $h = \sqrt{4-w^2}$: the diagonal is $\sqrt{w^2+h^2} = 2$, so the diagonal chain (Lemma 5.2) gives $N_{\max}(w, h) \geq \lfloor 2 \rfloor + 1 = 3$. For $\xi \leq w \leq 1/\sqrt{2}$ and $h = 2\sqrt{2} - 2w$: give all three squares the diagonal side directions $e_1 = (1, 1)/\sqrt{2}$ and $e_2 = (-1, 1)/\sqrt{2}$, with centers $A = (0, 0)$, $B = (w,\ \sqrt{2} - w)$, $C = (0,\ 2\sqrt{2} - 2w)$; these lie in $[0, w] \times [0, 2\sqrt{2} - 2w]$, since $0 \leq \sqrt{2} - w \leq 2\sqrt{2} - 2w$. The consecutive differences $B - A = (w,\ \sqrt{2} - w)$ and $C - B = (-w,\ \sqrt{2} - w)$ have projections $(w + \sqrt{2} - w)/\sqrt{2} = 1$ on e_1 and on e_2 respectively, and the outer difference $C - A = (0,\ 2\sqrt{2} - 2w)$ has projection $2 - \sqrt{2}\,w \geq 1$ on each e_i, since $w \leq 1/\sqrt{2}$. Each pair of centers is separated by at least 1 along a common side axis, so the squares are interior-disjoint by Lemma 2.3 and $N_{\max}(w,\ 2\sqrt{2} - 2w) \geq 3$. For $1/\sqrt{2} \leq w < 1$ and $h = 1/w$: Proposition 6.1 with $r = 1$.

Thus for every $0 \leq w < 1$, three centers fit in $[0, w] \times [0, h]$ if and only if $h \geq T_3(w)$, and the infimum $T_3(w) = \inf\{h : N_{\max}(w, h) \geq 3\}$ is attained in every branch. □

6.4 Proof of Theorem 1.6

Fix $1/\sqrt{2} \le w < 1$ and $d = \sqrt{1 - w^2}$, and recall the thresholds $\tau_1 = 0$, $\tau_{2r+1} = r/w$ for $r \ge 1$, and $\tau_{2r+2} = r/w + d$ for $r \ge 0$. The sequence (τ_n) is strictly increasing, since $0 < d < 1/w$, so the bands $[\tau_n, \tau_{n+1})$ partition $[0, \infty)$.

Proof of Theorem 1.6. Upper bounds: $h < \tau_m$ forbids m centers. Let $m \ge 2$ and suppose m interior-disjoint unit squares have centers in $[0, w] \times [0, h]$; order the centers by height, with consecutive vertical gaps $g_1, \dots, g_{m-1} \ge 0$. Any two centers differ horizontally by at most $w \le 1$, so $g_i \ge d$ by Corollary 2.2. Each consecutive triple of centers lies in a translate of $[0, w] \times [0, g_i + g_{i+1}]$, so Theorem 6.5 forces $g_i + g_{i+1} \ge 1/w$; this is where the hypothesis $w \ge 1/\sqrt{2}$ enters. Set $u_i = g_i - d \ge 0$ and $c_0 = 1/w - 2d$, so that $u_i + u_{i+1} \ge c_0$ for each i, and $c_0 \ge 0$ since $2wd = 2w\sqrt{1 - w^2} \le 1$. Summing $u_i + u_{i+1} \ge c_0$ over the $\lfloor (m-1)/2 \rfloor$ disjoint gap pairs $(g_1, g_2), (g_3, g_4), \dots$, and discarding the leftover excess $u_{m-1} \ge 0$ when $m - 1$ is odd, gives

$$h \;\ge\; \sum_{i=1}^{m-1} g_i \;=\; (m-1)d + \sum_{i=1}^{m-1} u_i \;\ge\; (m-1)d + \left\lfloor \frac{m-1}{2} \right\rfloor \left(\frac{1}{w} - 2d \right) \;=\; \tau_m.$$

Indeed, at $m = 2r + 1$ the right-hand side is $2rd + r(1/w - 2d) = r/w = \tau_{2r+1}$, and at $m = 2r + 2$ it is $(2r+1)d + r(1/w - 2d) = r/w + d = \tau_{2r+2}$. Hence $h < \tau_m$ implies $N_{\max}(w, h) \le m - 1$.

Lower bounds: $N_{\max}(w, \tau_n) \ge n$. For $n = 1$ this is trivial. For $n = 2$ and $\tau_2 = d$: the shared-edge pair of Corollary 2.2, with common side direction (w, d) and centers $(0, 0)$ and (w, d), sits at opposite corners of $[0, w] \times [0, d]$. For $n = 2r + 1$ with $r \ge 1$: Proposition 6.1 places $2r + 1$ centers at height $r/w = \tau_{2r+1}$. For $n = 2r + 2$ with $r \ge 1$: Proposition 6.2 places $2r + 2$ centers at height $r/w + d = \tau_{2r+2}$.

For $\tau_n \le h < \tau_{n+1}$, the lower bound and monotonicity of $N_{\max}(w, \cdot)$ give $N_{\max}(w, h) \ge n$, while the upper bound with $m = n + 1$ gives $N_{\max}(w, h) \le n$. □

Remark 6.6. The odd thresholds and every band's lower half are elementary, resting only on the constructions of Section 6.1 and Corollary 2.2. Each band's upper half for $n \ge 2$ consumes Theorem 6.5, and hence Theorem 6.3, through the consecutive-triple estimate, so only the $n = 1$ band is fully elementary. With the center-area lemma proved in [7] all bands are unconditional, and the band upper bounds are formalized in Lean on top of the formalized center-area lemma (Section 9).

6.5 The wide branch recovers the center-area lemma

Proposition 6.7. *Assume the two narrow branches of Theorem 1.5 together with the wide-branch obstruction: if $1/\sqrt{2} \le w < 1$ and three centers fit in $[0, w] \times [0, h]$, then $h \ge 1/w$. Then Theorem 6.3 follows.*

Proof. We first show that a non-obtuse triangle with long sides and a side of length at least $\sqrt{2}$ has area at least $1/2$, then trap a putative counterexample in a thin rectangle and rule it out branch by branch.

Step 1: a triangle with all sides at least 1, longest side $c \ge \sqrt{2}$, and no obtuse angle has area at least $1/2$. Let r be the altitude onto the longest side. The base angles are non-obtuse, so the foot of the altitude lies in the base, splitting $c = u + v$ with $u, v \ge 0$; the other two sides give $u^2 + r^2 \ge 1$ and $v^2 + r^2 \ge 1$. In coordinates with the apex at the origin and the base horizontal, the vectors from the apex to the base endpoints are $(-u, -r)$ and $(v, -r)$, so non-obtuseness at the apex gives $r^2 \ge uv$. If $r^2 < 1/2$, then $u^2, v^2 > 1/2$, whence $uv > 1/2 > r^2$, a contradiction; so $r \ge 1/\sqrt{2}$ and the area is $cr/2 \ge \sqrt{2} \cdot (1/\sqrt{2})/2 = 1/2$.

Step 2: a counterexample lies in a thin rectangle. Suppose Theorem 6.3 fails: three interior-disjoint unit squares whose centers form a non-obtuse triangle of area less than $1/2$. All sides are at least 1 (Lemma 2.1), and by Step 1 the longest side satisfies $c < \sqrt{2}$, so $c \in [1, \sqrt{2})$. Let r be the altitude onto the longest side; its foot again lies in the base, so the three centers lie in an $r \times c$ rectangle with $rc = 2\,\text{area} < 1$; in particular $r < 1/c \le 1 \le c$.

Step 3: both branches refute it. After a rigid motion of the configuration, the three centers lie in $[0, r] \times [0, c]$ with $r < 1$, so three centers fit there. If $r < 1/\sqrt{2}$, the narrow branches give $c \ge T_3(r) \ge \sqrt{2}$, since $\sqrt{4 - r^2} \ge \sqrt{2}$ on $[0, \xi]$ and $2\sqrt{2} - 2r \ge 2\sqrt{2} - \sqrt{2} = \sqrt{2}$ on $[\xi, 1/\sqrt{2}]$ — contradicting $c < \sqrt{2}$. If $1/\sqrt{2} \le r < 1$, the wide-branch obstruction gives $c \ge 1/r$, contradicting $rc < 1$. $\square$

The wide branch of Theorem 1.5 is therefore not merely the easiest route to its third case: given the narrow branches, it is equivalent to the center-area input, the $1/w$ law being the center-area phenomenon in another form.

7 A toolbox of sharp local bounds

The results of this section arose as candidate strengthenings of the tools above; each is sharp, several in the deflating sense that the obvious estimate cannot be improved. Throughout, $q(t) = (\cos t + \sin t)/2$ is the support radius of (2), and $(t)_+ = \max\{t, 0\}$.

7.1 The exact near-wall gap

Consider two interior-disjoint unit squares packed against a wall: both lie in the half-plane $\{Y \ge 0\}$, with centers at heights $y_1 \le y_2$. Containment forces $y_i \ge 1/2$, and for $1/2 \le y \le 1/\sqrt{2}$ the frame of a wall-feasible square can deviate from axis alignment by at most

$$\alpha(y) \;=\; \arcsin\bigl(\sqrt{2}\,y\bigr) - \frac{\pi}{4},$$

the angle determined by $q(\alpha(y)) = y$; for $y \ge 1/\sqrt{2}$ every orientation is feasible, and we set $\alpha(y) = \pi/4$ there. Let $\Gamma(y_1, y_2)$ be the infimum of the horizontal center gap $|x_1 - x_2|$ over such pairs. A direction $(\cos\varphi, \sin\varphi)$ with $0 \le \varphi < \pi/2$ makes angle $\min\{\varphi, \pi/2 - \varphi\}$ with the nearest side normal of the axis-parallel frame, and rotating the frame by up to $\alpha(y)$ toward the direction leaves residual angle $(\min\{\varphi, \pi/2 - \varphi\} - \alpha(y))_+$; so the least support of a wall-feasible square at height y in that direction is $m_y(\varphi) = q\bigl((\min\{\varphi, \pi/2 - \varphi\} - \alpha(y))_+\bigr)$, and

$$\Gamma(y_1, y_2) \;=\; \max\Bigl\{0,\ \inf_{0 \le \varphi < \pi/2} \frac{m_{y_1}(\varphi) + m_{y_2}(\varphi) - (y_2 - y_1)\sin\varphi}{\cos\varphi}\Bigr\}. \tag{11}$$

Indeed, interior-disjoint convex bodies are weakly separated by a line, whose unit normal may be taken as $(\cos\varphi, \sin\varphi)$ after reflections, and projecting the center displacement onto that normal gives $|x_1 - x_2| \cos\varphi \ge m_{y_1}(\varphi) + m_{y_2}(\varphi) - (y_2 - y_1)\sin\varphi$. Conversely, a common frame attaining both least supports at the minimizing φ, displaced horizontally so that the two supporting lines coincide, realizes the infimum.

Theorem 7.1 (Exact shallow-wall gap). *Let $1/2 \le y_1 \le y_2 \le 1/\sqrt{2}$ and $\delta = y_2 - y_1$, and put*

$$s \;=\; y_1 - \sqrt{\tfrac{1}{2} - y_1^2}, \qquad c \;=\; y_1 + \sqrt{\tfrac{1}{2} - y_1^2},$$

so that $s = \sin\alpha(y_1)$, $c = \cos\alpha(y_1)$. Then

$$\Gamma(y_1, y_2) \;=\; \begin{cases} \sqrt{1 - \delta^2}, & \delta \le s, \\ \dfrac{1 - \delta s}{c}, & \delta \ge s, \end{cases}$$

attained by giving both squares the common frame angle $\min\{\alpha(y_1), \arcsin\delta\}$. *The wall improves on the free incircle bound* $\sqrt{1-\delta^2}$ *exactly when* $\delta > s$:

$$\Big(\frac{1-\delta s}{c}\Big)^2 - (1-\delta^2) \;=\; \Big(\frac{\delta - s}{c}\Big)^2.$$

Proof. Write $a_i = \alpha(y_i)$, so $a_1 \le a_2 \le \pi/4$ and $\delta \le 1/\sqrt{2} - 1/2$; we minimize the objective of (11) on the four subintervals cut by a_1, a_2, $\pi/4$.

On $[0, a_1]$ both least supports equal 1/2 and the objective is $\sec\varphi - \delta\tan\varphi$, with derivative $(\sin\varphi - \delta)/\cos^2\varphi$; its minimum is at $\varphi = \arcsin\delta$ when $\delta \le s = \sin a_1$, with value $\sqrt{1-\delta^2}$, and at $\varphi = a_1$ when $\delta \ge s$, with value $(1-\delta s)/c$.

On $[a_1, a_2]$ the addition formulas and $c+s = 2y_1$ give $m_{y_1}(\varphi) = q(\varphi - a_1) = A_1\cos\varphi + y_1\sin\varphi$ with $A_1 = (c-s)/2$, so the objective is $A_1 + (y_1 - \delta)\tan\varphi + \frac{1}{2}\sec\varphi$, increasing because $y_1 - \delta = 2y_1 - y_2 \ge 1 - 1/\sqrt{2} > 0$.

On $[a_2, \pi/4]$ the objective is likewise $A_1 + A_2 + (y_1 + y_2 - \delta)\tan\varphi$ with $A_2 = (\cos a_2 - \sin a_2)/2$, increasing because $y_1 + y_2 - \delta = 2y_1 > 0$.

For $\varphi \in [\pi/4, \pi/2)$ the two supports sum to at least 1, and $(1 - \delta\sin\varphi)/\cos\varphi \ge 1$ because $\tan(\varphi/2) \ge \tan(\pi/8) > 1/\sqrt{2} - 1/2 \ge \delta$; this never beats the value 1 at $\varphi = 0$.

The objective is continuous, equal to $(1-\delta s)/c$ at a_1 from either side, so the infimum is the branch value stated; the displayed identity, a direct computation from $c^2 + s^2 = 1$, confirms $(1-\delta s)/c \ge \sqrt{1-\delta^2}$, so each branch minimum is global on its range of δ. The frame at angle $\varphi_0 = \min\{a_1, \arcsin\delta\}$ is feasible for both squares ($\varphi_0 \le a_1 \le a_2$) and attains both least supports at the minimizing angle, so the common-frame configuration of (11) attains the infimum. □

Remark 7.2 (Optimized wall-band gap). For $1/2 \le h \le 1/\sqrt{2}$ let $\gamma(h) = \min_{1/2 \le y_1 \le y_2 \le h}\Gamma(y_1, y_2)$, the least gap over a wall band of height h. Both branches of Theorem 7.1 decrease in y_2, so $y_2 = h$; parametrizing $y_1 = q(a)$ with $s = \sin a$, $c = \cos a = \sqrt{1-s^2}$ and optimizing the constrained branch $(1-\delta s)/c$, the minimum occurs at the unique root of

$$h \;=\; 2s + \frac{(1-s^2)^{3/2} - s^3}{2}, \qquad \text{giving} \qquad \gamma(h) \;=\; c - \frac{s^2(c-s)}{2}.$$

At $h = 0.7$ the optimal $s = 0.10435686\ldots$ is the root in $(0.104, 0.105)$ of the sextic

$$50s^6 - 275s^4 + 70s^3 + 475s^2 - 280s + 24 \;=\; 0,$$

and $\gamma(0.7) = 0.98969271\ldots$; these constants are machine-verified, while the uniqueness and branch-membership claims above are stated with compressed justification. Even the free bound already prunes on this band: $\delta \le 0.2$, so $\Gamma \ge \sqrt{0.96} > 0.9797$.

7.2 The corner threshold

Theorem 7.3 (Corner threshold)**.** *Let* $C_* = 1 + \frac{1}{2\sqrt{2}}$. *Among unit squares contained in the quarter plane* $[0,\infty)^2$ *with pairwise disjoint interiors, at most one can have its center in* $[0, C]^2$ *for any* $C < C_*$; *and at* $C = C_*$ *two such centers are possible.*

Proof. Suppose two squares contained in $[0,\infty)^2$ with disjoint interiors have centers $c_A, c_B \in [0, C]^2$. Write $f(t) = \cos t + \sin t$, let $\alpha_A, \alpha_B \in [0, \pi/4]$ be their frame deviations from the axes, and put $\sigma_i = f(\alpha_i)$; containment forces both coordinates of each center to be at least $\sigma_i/2$, the square's support $q(\alpha_i)$ in the coordinate directions. By the separating axis theorem some unit side normal u of one square, say A, is a separating direction; then $\|u\|_1 = \sigma_A$, the support of A in direction u is 1/2, and the support of B is $f(\delta)/2$, where $\delta \in [0, \pi/4]$ is the frame distance. Both coordinates of each center lie in $[\min(\sigma_A, \sigma_B)/2, C]$, so with $H(\delta) = (1 + f(\delta))/2$,

$$H(\delta) \;\le\; |(c_B - c_A)\cdot u| \;\le\; \|u\|_1\, \|c_B - c_A\|_\infty \;\le\; \sigma_A\Big(C - \frac{\min(\sigma_A, \sigma_B)}{2}\Big),$$

hence $C \geq \min(\sigma_A, \sigma_B)/2 + H(\delta)/\sigma_A$.

If $\sigma_A \leq \sigma_B$, then $H(\delta) \geq 1$ and $C \geq \sigma_A/2 + 1/\sigma_A \geq \sqrt{2} > C_*$ by AM–GM. Otherwise $\alpha := \alpha_A > \beta := \alpha_B$; the frame distance is $\alpha - \beta$ or $\min\{\alpha+\beta, \pi/2 - \alpha - \beta\}$, each at least $\alpha - \beta$ (using $\alpha \leq \pi/4$), and f increases on $[0, \pi/4]$, so $H(\delta) \geq (1 + f(\alpha - \beta))/2$ and

$$C \;\geq\; F(\alpha, \beta) \;:=\; \frac{f(\beta)}{2} + \frac{1 + f(\alpha-\beta)}{2f(\alpha)}.$$

For fixed α, $\partial^2(2F)/\partial\beta^2 = -f(\beta) - f(\alpha-\beta)/f(\alpha) < 0$, so F is concave in β and minimized at an endpoint of $[0, \alpha]$:

$$F(\alpha, 0) = 1 + \frac{1}{2f(\alpha)} \;\geq\; 1 + \frac{1}{2\sqrt{2}} = C_*, \qquad F(\alpha, \alpha) = \frac{f(\alpha)}{2} + \frac{1}{f(\alpha)} \;\geq\; \sqrt{2},$$

using $f(\alpha) \leq \sqrt{2}$ and AM–GM. In every case $C \geq C_*$, which proves the bound.

At $C = C_*$, take the axis-parallel square centered at $(1/2, 1/2)$ and the square at frame angle $\pi/4$ centered at (C_*, C_*), both contained in the quarter plane. The center displacement projects onto the diagonal direction to $(C_* - \frac{1}{2})\sqrt{2} = (\sqrt{2}+1)/2$, exactly the sum $\sqrt{2}/2 + 1/2$ of the two supports in that direction, so the squares touch along a common supporting line and have disjoint interiors. ☐

7.3 Exact gap, angle bound, and three failures

Corollary 2.2 forces two squares at vertical center gap $d \in [0, 1]$ apart horizontally by $\sqrt{1-d^2}$, which is exactly what their incircles already force. One might hope the squares themselves force more; they do not.

Proposition 7.4 (Exact horizontal gap)**.** *For $d \in [0,1]$ let $g(d)$ be the infimum of $|x_1 - x_2|$ over pairs of interior-disjoint unit squares with centers (x_1, y_1) and (x_2, y_2) satisfying $|y_1 - y_2| = d$. Then $g(d) = \sqrt{1-d^2}$, and the infimum is attained.*

Proof. The lower bound is Corollary 2.2 with the coordinate roles exchanged. The shared-edge pair of that corollary with common side direction $(\sqrt{1-d^2}, d)$ has $|y_1 - y_2| = d$ and $|x_1 - x_2| = \sqrt{1-d^2}$. ☐

Any improvement on the incircle contraction must therefore retain orientation or contact information beyond the two coordinate gaps.

Proposition 7.5 (Quantitative angle bound)**.** *Let T be the triangle on the centers of three interior-disjoint unit squares and γ its angle at a specified vertex. Then* $\operatorname{area}(T) \geq \frac{1}{2}\sin\gamma$.

Proof. The two sides adjacent to the specified vertex have lengths $a, b \geq 1$ by Lemma 2.1, so $\operatorname{area}(T) = \frac{1}{2}ab\sin\gamma \geq \frac{1}{2}\sin\gamma$; collinear triples give $0 \geq 0$. ☐

This degrades gracefully from the non-obtuse value $1/2$ of Theorem 6.3 as $\gamma \to \pi$, which is what a branch-and-bound prover needs when non-obtuseness cannot be certified; by contrast, the next three natural strengthenings all fail.

Proposition 7.6 (Four-center hull infimum is zero)**.** *For every $\varepsilon > 0$ there are four pairwise disjoint axis-parallel unit squares whose centers are in strictly convex position with convex-hull area less than ε.*

Proof. Fix $L > 1$ and $\eta > 0$, and center axis-parallel unit squares at

$$p_0 = (0,0), \quad p_1 = (L, \eta), \quad p_2 = (2L, -\eta), \quad p_3 = (3L, 0).$$

The closed horizontal projections $[x_i - \frac{1}{2}, x_i + \frac{1}{2}]$ are pairwise disjoint, so the squares are pairwise disjoint. In the cyclic order p_0, p_2, p_3, p_1 every turn has cross product $3L\eta > 0$, so the centers are in strictly convex position, and the shoelace formula gives hull area $3L\eta$, less than ε for η small. ☐

No hull-area analogue of the center-area lemma holds for four centers, even for axis-parallel squares: non-obtuseness does essential work at three centers.

Remark 7.7 (No uniform second-layer gap). One might hope that a square in the wall band ($1/2 \le y \le 0.7$) and a square in a second layer above it (center height at least 1.2) are forced apart horizontally by a positive amount. They are not: the axis-parallel squares centered at $(1/2, 1/2)$ and $(1/2, 3/2)$ satisfy the layer constraints and touch edge to edge with horizontal gap 0. (For center gaps $\delta < 1$ an exact pointwise formula follows from (11) with the upper square unconstrained; it is uniformity over the layers that fails.)

Proposition 7.8 (No chord-distribution surplus)**.** *Let $0 < D < 1$, let S be a unit square whose sides make angle $\theta \in (0, \pi/4]$ with the axes (available by the square's symmetries; $\theta = 0$ is incompatible with the hypothesis below), and write $c = \cos\theta$, $s = \sin\theta$. Let ℓ_1, ℓ_2 be vertical lines at distance D from one another with the center of S between them, at distances t_1, t_2 from the center ($t_1 + t_2 = D$), and suppose each line cuts a corner triangle off S: $(c-s)/2 \le t_i \le (c+s)/2$. Then the chord lengths $l_i = \mathrm{length}(\ell_i \cap S)$ satisfy*

$$l_1 + l_2 \;\ge\; 2(\sqrt{2} - D) \qquad \textit{and} \qquad \max\{l_1, l_2\} \;\ge\; \sqrt{2} - D,$$

both attained simultaneously by the square with $\theta = \pi/4$ centered midway between the lines.

Proof. Put $p = c + s \in (1, \sqrt{2}]$, so $cs = (p^2 - 1)/2$. In the coordinates with the center of S at the origin, a vertical line at corner-cutting distance t crosses the two sides through the extreme vertex $(p/2, (s-c)/2)$, at heights differing by $(p/2 - t)(c/s + s/c) = (p/2 - t)/(cs)$; the hypothesis on t places both crossings on those sides. Summing over the two lines,

$$l_1 + l_2 \;=\; \frac{p - D}{cs} \;=\; \frac{2(p - D)}{p^2 - 1},$$

whose p-derivative is $-2\big((p - D)^2 + 1 - D^2\big)/(p^2 - 1)^2 < 0$; so the sum is minimized over orientations at $p = \sqrt{2}$, with value $2(\sqrt{2} - D)$, and the larger chord is at least half of it. The square with $\theta = \pi/4$ centered midway between the lines has $t_1 = t_2 = D/2$ and chords $(\sqrt{2}/2 - D/2)/(1/2) = \sqrt{2} - D$, attaining both bounds. □

For $D \ge 1/\sqrt{2}$, sliding that square between the lines realizes every split of the fixed total $2(\sqrt{2} - D)$ between l_1 and l_2, so nothing more can be extracted from the pair of chord lengths alone.

8 Open problems

Three questions remain open.

Question 8.1. Determine T_4, the exact threshold at which a fourth center fits, for all $w < 1$, and more generally the full band structure for $w < 1/\sqrt{2}$. Theorem 1.6 covers $1/\sqrt{2} \le w < 1$, and Theorem 1.3 covers everything for $w \le \tau$.

Question 8.2. Is $N_{\max}(a, b) = (\lfloor a \rfloor + 1)(\lfloor b \rfloor + 1)$ whenever $a - \lfloor a \rfloor$ and $b - \lfloor b \rfloor$ are sufficiently small? Theorem 1.2 gives the upper bound $(a+1)(b+1)$; the question is whether the integer-grid maximum is locally rigid.

Question 8.3. For which convex K is the bound of Theorem 1.1 within an additive constant of sharp? The proof loses nothing for integer rectangles; quantifying the slack for tilted strips would sharpen its use as a pruning rule.

9 Lean validation

Two of the paper's main results are machine-checked end to end in Lean 4 over mathlib [8, 6]. The source repository carries a small Lean library, `PaperProofs`, of wrapper theorems restating those results in the forms used in the paper, together with the underlying formal development.

Formalized. The rectangle center bound (Theorem 1.2) is `SquareFacts.CenterBound.card_le_of_rotated_unit_square_centers`: for any finite index type, centers in the rectangle, unit direction vectors, and pairwise-disjoint open rotated squares, $\#\iota \le (a+1)(b+1)$. The development proves the one-square inequality (Lemma 3.2) in centered coordinates as a product-measure statement (`RotatedSquare`, `QuadrantScore`, `RectangleScore`, and supporting files, about 4,600 lines), and the integer-grid sharpness construction separately (`Sharpness`). The wrapper `PaperProofs.Nmax.paper_rectangle_center_bound` restates the bound, and `PaperProofs.Nmax.paper_nmax_rectangle_integer` assembles bound and sharpness into the exact value $N_{\max}(a,b) = (a+1)(b+1)$ for $a, b \in \mathbb{N}$, in the project's finite-packing presentation (an explicit witness family together with the finite-family upper bound). The exact very-narrow strip law (Theorem 1.3) is `SquareFacts.WideStrip.nmaxValue_very_narrow_strip`, restated as `PaperProofs.Nmax.paper_very_narrow_strip_law`. The band upper bounds of Theorem 1.6 are formalized on top of a formalized proof of the center-area lemma (companion development [7]); the bridge theorem `axisAlignedCentralCompactBranch_from_imported` of the namespace `SquareFacts.ElementaryObstructions` (file `CenterAreaBridge`) imports `CenterAreaLemma.center_area_lemma` and discharges the geometric hypothesis, so those results are also unconditional in Lean.

Not formalized. The convex Minkowski bound (Theorem 1.1), the cubical clipping lemma (Lemma 4.1) behind it, and the toolbox of Section 7 are supported by the prose proofs and by independent numerical audit only.

Build status. The complete development (the formal libraries and the wrappers) was rebuilt end to end on 2026-08-09 with Lean v4.29.1 and mathlib v4.29.1 (8,367 build jobs). It contains no `sorry`, and `#print axioms` on the wrapper theorems reports only the standard classical axioms (`propext`, `Classical.choice`, `Quot.sound`).

Acknowledgments

This paper belongs to a larger machine-assisted project on square packing, and substantial computer assistance went into its production. Large-language-model agents carried out the search for proofs and counterexamples, drafted the arguments, and wrote the Lean 4 formalizations over mathlib [8, 6] of Theorem 1.2, Theorem 1.3, the band upper bounds of Theorem 1.6, and the center-area lemma [7] on which those bounds depend. Every closed-form constant and inequality in the paper was subjected to an automated numerical audit, and every argument was also checked by hand. Any remaining errors are the author's responsibility.

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